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PRACTICAL MATHEMATICS

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(STAGE I.)

BY

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PREFACE TO FIRST EDITION

THE object of this book is to develop in the student a clear and accurate conception of the more useful principles of Elementary Mathematics. For this purpose it is certainly not necessary that the student should master the complex scheme of rigid argument from which these principles are ultimately deduced; for example, the Sixth Book of Euclid is in no way essential to an accurate practical knowledge of the properties of Similar Figures.

A large number of worked examples have been inserted, and the book is well supplied with examples for practice. Both have been chosen with a view to elucidate principles, and to train the intelligence.

The chapters on Algebra are intended to give the student a thorough grasp of the meaning and use of Algebraical Symbols, Formulæ, and Equations, including equations of Variation. The theory of Indices has been explained with a view to its use in Logarithms. More complex algebraical operations, such as the manipulation of difficult fractions, have been omitted.

In Geometry, both Plane and Solid, every effort has been made to appeal directly to the sense of shape and measurement, and to give suitable practice in the use of Mathematical Instruments for geometrical calculation. In Descriptive Geometry it is especially necessary to warn the student not to learn rules by rote, but to remember them in connection with the solid figures to which they are intended to apply.

Sufficient Trigonometry has been introduced for the application of Elementary Trigonometrical Formulæ.

Special care has been devoted to the chapters on the Use of Squared Paper, and on Vectors, which should furnish useful training for practical purposes.

In conclusion, I wish to express my thanks to my friend Mr. J. H. Dibb, B.Sc., for many useful suggestions in the course of the work.

A. G. CRACKNELL.

October 27, 1900.

NOTE TO THE FOURTH EDITION

In this edition some new Examination Papers have been added. I wish to express my thanks to the Controller of His Majesty's Stationery Office for the kind permission to include the papers set by the Board of Education.

July 25, 1906. A. G. C.

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PRACTICAL MATHEMATICS

CHAPTER I.

DECIMAL FRACTIONS.

1. The meaning of the decimal notation.—The quantity ·32587 is a "**decimal fraction**;" the dot placed in front of the figures, and nearly or quite on a level with the top of the figures, is called the "**decimal point**;" the figure in the "**first decimal place**" (in this case 3) denotes **tenths** of the unit; the figure in the second decimal place (in this case 2) denotes hundredths; the next figure denotes thousandths, the next ten-thousandths, and so on.

Thus the symbol ·32587 denotes $\frac{3}{10} + \frac{2}{100} + \frac{5}{1000} + \frac{8}{10000} + \frac{7}{100000}$; and the symbol ·00301 denotes $\frac{0}{10} + \frac{0}{100} + \frac{3}{1000} + \frac{0}{10000} + \frac{1}{100000}$; *i.e.* $\frac{3}{1000} + \frac{1}{100000}$.

If we wish to represent a whole number and a fraction, we simply write the whole number in front of the decimal point; thus 24·23 denotes $24 + \frac{2}{10} + \frac{3}{100}$.

Note that the correct way to read the symbol ·324 is "**decimal three two four**" (or "point three two four"), **not** "decimal three hundred and twenty-four."

Note that if we place the figure 0 at the end of a decimal fraction, we do not alter its value; *e.g.*—

$$\cdot 32 = \frac{3}{10} + \frac{2}{100}$$
$$\cdot 320 = \frac{3}{10} + \frac{2}{100} + \frac{0}{1000}$$
$$= \frac{3}{10} + \frac{2}{100}$$

2. To convert a decimal into a vulgar fraction.

RULE.—**In the numerator place the figures of the decimal, without the decimal point. In the denominator place the figure 1, followed by the figure 0 repeated as many times as there are figures after the decimal point.**

EXAMPLES (1).—*Convert* ·321 *to a vulgar fraction.*

Following the rule, the numerator will be 321; and, since there are three figures after the decimal point, the denominator will be 1 followed by three 0's, *i.e.* 1000; hence we obtain $\frac{321}{1000}$.

(2).—$\cdot 00507 = \frac{00507}{100000} = \frac{507}{100000}$.

(3).—$23\cdot 2157 = \frac{232157}{10000}$.

(4).—$\cdot 125 = \frac{125}{1000} = \frac{25}{200} = \frac{1}{8}$.

Explanation.—The reason for the rule is obvious if we work on the principle of the preceding paragraph;

e.g.

$$\cdot 321 = \frac{3}{10} + \frac{2}{100} + \frac{1}{1000}$$
$$= \frac{300 + 20 + 1}{1000} = \frac{321}{1000}$$

Again

$$23\cdot 2157 = 23 + \frac{2}{10} + \frac{1}{100} + \frac{5}{1000} + \frac{7}{10000}$$
$$= \frac{230000 + 2000 + 100 + 50 + 7}{10000}$$
$$= \frac{232157}{10000}$$

Examples.—I.

Reduce to the equivalent vulgar fraction or mixed number—

1. ·5; ·25; ·75; ·125; ·375; ·625; ·875.
2. ·112; ·264; ·3125; 5·2087; 17·004.
3. 20·264; 135·15625; ·0064; 4·0000256.
4. Learn by heart the results obtained in Question 1, as they are constantly occurring.

3. Addition and subtraction of decimal fractions.

Rule.—**Proceed as in ordinary addition, except that the quantities to be added or subtracted must be placed so that the decimal points are in the same vertical line.**

Note that this arrangement sets the figures in the first decimal place in one column, the figures in the second decimal place in another column, and so on. This is obvious from the following examples.

Example (1).—*Add together* 24·2, 3·5715, ·283, ·00245, 28·0013.

```
24·2
 3·5715
  ·283
  ·00245
28·0013
-------
56·05825
```

Explanation.—In the column which contains figures in the fifth decimal place there is only one figure, the 5 in the fourth line; as there is nothing to add to this, we place it in the fifth decimal place in the answer. In the column which contains figures in the fourth decimal place we have three figures, 5, 4, and 3; each of these represents ten-thousandths, giving a total of 12 ten-thousandths; but remembering that 10 ten-thousandths = 1 thousandth, it follows that 12 ten-thousandths = 1 thousandth + 2 ten-thousandths. The 2 ten-thousandths are placed in the answer as 2 in the fourth decimal place, and the 1 thousandth is "carried," *i.e.* added on to the other thousandths represented in the column of figures in the third decimal place. Thus the process is exactly analogous to ordinary addition, provided the decimal points are placed in one vertical line.

EXAMPLE (2).—*Subtract* 34·987624 *from* 210·0992.

```
210·0992
 34·987624
──────────
175·111576
```

EXPLANATION.—The last two figures in the subtrahend (the second line) represent 2 hundred-thousandths and 4 millionths; hence together they represent 24 millionths; since there are no numbers directly above them from which to subtract them, we "borrow" 1 ten-thousandth from the figure in the fourth decimal place in the upper line; now, 1 ten-thousandth = 100 millionths; subtracting from this the 24 millionths from the second line, we have 76 millionths left, which appear in the result as 7 in the fifth decimal place and 6 in the sixth decimal place; the 1 ten-thousandth which was "borrowed" is "paid back" to the fourth decimal place, in accordance with the usual arithmetical rule.

EXAMPLES.—II.

1. Add together ·00234, 2·24, 31·208, ·00025, 216·3.
2. Add together 428·2681, 3456·7, 2·00345, 71·231, ·345612, 123·1234, 183.*
3. From 123·456 subtract 45·678.
4. From 987·6543 subtract 98·76543; and from the result subtract 9·876543.
5. Find the difference between 3425·16 and 61·5243.
6. Subtract 234·128 from 1,000,000.

4. To multiply or divide a decimal by 10, 100, 1000, etc.

RULE.—

To **multiply** by	**10,**	move the	decimal	point	**one** place	to the right.
„	**100,**	„	„		**two** places	„
„	**1000,**	„	„		**three** „	„
„	**10,000,**	„	„		**four** „	„

and so on.

To **divide** by	**10,**	move the	decimal	point	**one** place	to the left.
„	**100,**	„	„		**two** places	„
„	**1000,**	„	„		**three** „	„

and so on.

---

* 183 must be placed so that the 3 falls in the units' column, *i.e.* the column immediately in front of the decimal points.

EXPLANATION.—Take any decimal quantity, *e.g.* 324·576 ; if we move the point one place to the right, we obtain 3245·76. Now, it is obvious that the *value of each figure has been increased tenfold ; e.g.* the 2 represents 2 tens in the first case, and 2 hundreds in the second case ; the 5 represents 5 tenths in the first case, and 5 units in the second case ; the 6 represents 6 thousandths in the first case, and 6 hundredths in the second case. It follows that when the point is moved one place to the right, *the new quantity is ten times as great as the original quantity.*

It also follows that to move the point two places to the right is equivalent to multiplying by 10 twice, *i.e.* is equivalent to multiplying by 100. Also that to move the point one place to the left *diminishes the value of each figure to one-tenth of its former value,* and is therefore equivalent to dividing the whole quantity by 10 ; and so on.

EXAMPLES.—

$$23{\cdot}456 \times 10 = 234{\cdot}56$$
$${\cdot}0234 \times 1000 = 023{\cdot}4 = 23{\cdot}4$$
$$234{\cdot}28 \div 100 = 2{\cdot}3428$$
$$234{\cdot}28 \div 10{,}000 = {\cdot}023428$$
$$4{\cdot}26 \times 10{,}000 = 42{,}600{\cdot}0 = 42{,}600$$

Note that in the last two cases we have to fill in with cyphers in order to be able to move the decimal point through the required number of places.

5. Multiplication of decimals.—This follows the same rules as ordinary multiplication ; but the position of the decimal point in the product is determined by the following

RULE.—**The number of decimal places in the product is equal to the sum of the numbers of decimal places in the multiplier and multiplicand.**

EXAMPLE (1).—*Multiply* 5·3248 *by* 20·214. There are four decimal places in the multiplicand, and three in the multiplier ; therefore by the above rule there must be seven decimal places in the product ; hence, after finishing the multiplication we mark off seven decimal places in the result, as follows :—

```
     5·3248
     20·214
     ------
     212992
     53248
   106496
 106496
 ---------
 107·6355072
```

EXPLANATION.—$5{\cdot}3248 = \frac{53248}{10000}$; also $20{\cdot}214 = \frac{20214}{1000}$. We may multiply these two vulgar fractions together by multiplying the two numerators together and the two denominators together. This gives $\frac{1076355072}{10000000}$, or $107\frac{6355072}{10000000}$, which is equivalent to 107·6355072.

EXAMPLE (2).—*Multiply* 1·248 *by* 23. There are three decimal places in the multiplicand, and none in the multiplier ; hence there are three decimal places in the product.

$$\begin{array}{r} 1{\cdot}248 \\ 23 \\ \hline 3744 \\ 2496 \\ \hline 28{\cdot}704 \end{array}$$

EXAMPLE (3).—*Multiply* ·00248 *by* ·042. The number of decimal places in the product = 5 + 3 = 8.

$$\begin{array}{r} {\cdot}00248 \\ {\cdot}042 \\ \hline 496 \\ 992 \\ \hline {\cdot}00010416 \end{array}$$

In order to mark off eight decimal places in the product, *we must prefix three cyphers to the five figures* which we obtain by multiplying.

EXAMPLES.—III.

Evaluate—

1. 348·236 × 100.	**2.** 248·316 ÷ 1000.	**3.** 2·1724 ÷ 100.
4. 368·2 × 1000.	**5.** 712·201 ÷ 10,000.	**6.** ·0002003 × 10,000.
7. 82·23 × 1·48.	**8.** 721·213 × ·00024.	**9.** ·00021 of ·00213.
10. 8·2134 × 240.	**11.** 2295 × ·0025.	**12.** ·12345 of ·0123.

6. Division of a decimal by a whole number.—This is exactly the same process as ordinary division ; the position of the decimal point in the quotient is determined by the following

RULE.—**The figure of the quotient obtained by the first step in the division must be put in the same position with regard to the decimal point as the last figure of the dividend which is used in the first step of the division.**

To master this rule, the following examples should be carefully studied :—

EXAMPLE (1).— $8\underline{)25\mathbf{2}{\cdot}312}$
31·539

Here the first step in the division is to divide 8 into **25**, which gives 3 and 1 over, and *since the* **5** *is in the tens' place in the dividend, we place the* 3 *in the tens' place in the quotient.*

EXAMPLE (2).— $7\underline{){\cdot}2\mathbf{5}585}$
·03655

Dividing 7 into **25** gives 3 and 4 over. *Since the* **5** *is in the second decimal place in the dividend,* we place the 3 *in the second decimal place* in the quotient ; *i.e.* we must put one cypher after the decimal point and before the 3.

EXAMPLE (3).—

```
325)12487·475(38·423
     975
     ----
     2737
     2600
     ----
     1374
     1300
     ----
      747
      650
      ---
      975
      975
```

Here the last figure of the dividend used in the first step of the division is the **8**, which is in the tens' place. Hence the 3 must be placed in the tens' place in the quotient, which requires that the decimal point should be placed before the 4 in the quotient.

EXAMPLE (4).—

```
325)1·573715(·0048422
     1300
     ----
      2737
      2600
      ----
       1371
       1300
       ----
         715
         650
         ---
          650
          650
```

Here the last figure in the dividend which is used in the first step in the division is the **3**, which is in the third decimal place. Hence the 4 in the quotient must be in the third decimal place, and must therefore have two cyphers placed in front of it.

EXPLANATION.—The reason for this rule is easily seen as follows: Taking, for instance, the last example, we start by dividing 1·573 by 325; but 1·573 = 1573 thousandths; and when we divide 1573 thousandths by 325, we obtain 4 thousandths, and a remainder of 273 thousandths; hence the 4 must be put in the third decimal place in the quotient.

7. Working to "seven significant figures."—In many cases we can continue the division process for a very large number of steps before it divides out exactly, *i.e.* before we get remainder 0 after subtraction. But in such cases it is of no practical importance to carry the process very far. Even in a very accurate scientific calculation it would be sufficient to carry such a division to *seven* steps only, and then leave it, ignoring both the remainder and the figures which would follow if the process were continued. This is called working to "*seven significant figures.*" *

For example—

```
23)32872·6(1429·243 . . .
   23
   --
    98
    92
    --
     67
     46
     --
     212
     207
     ---
       56
       46
       --
       100
        92
        --
         80
         69
         --
         11
```

If we continued the process, the next figure in the quotient would be 4, but being in the fourth decimal place it would only represent 4 ten-thousandths, a fraction which is quite insignificant compared to the quotient itself, which is over 1000.

Had the eighth figure in the quotient been 5 or more than 5, *we should have counted the seventh figure as* 4 *instead of* 3. This is done in order that our result containing 7 significant figures should be as near the actual quotient as possible.

Hence we obtain the following

RULE.—**Carry the division process to seven steps; if the next step would give a number in the quotient *greater* than 4, increase the seventh figure in the quotient by unity.**

EXAMPLES.—IV.

Evaluate—

1. ·3482 ÷ 8; 232·25 ÷ 5; ·003456 ÷ 9; 3249 ÷ 5; ·5 ÷ 8.

2. 247·11 ÷ 30 (in this case it will be better to divide by 10, using § 4, and then divide by 3); 20·01 ÷ 80; 20·01 ÷ 300; ·024 ÷ 8000; 26·1 ÷ 600; 22 ÷ 500.

3. 342·01 ÷ 23; ·023 ÷ 125; 2·242 ÷ 32; 13·5 ÷ 625; 112 ÷ 256; ·00234 ÷ 288.

4. Evaluate to seven significant figures: 222·28 ÷ 23; ·034 ÷ 126; ·281 ÷ 51; 11,248 ÷ 133; 236 ÷ 19; ·0004 ÷ 425.

---

* For the full meaning of this term, see § 28.

8. To divide by a decimal fraction.—This can be reduced to dividing by a whole number, by using the following

RULE.—**Make the divisor a whole number by moving the decimal point to the end; move the decimal point in the dividend the same number of places to the right as it was moved in the divisor; then proceed with the division as in § 6.**

EXAMPLE (1).—*Divide* 326·754 *by* 1·62.

To make the divisor 1·62 a whole number, we must move the decimal point two places to the right; if we move the decimal point in the dividend two places to the right, we obtain 32,675·4. Proceeding with the division—

```
162)32675·4(201·7
    324
    ---
      275
      162
      ---
      1134
      1134
```

EXAMPLE (2).—32,495·4 ÷ ·288.

To make the divisor a whole number, we move the point three places to the right; the dividend has only one decimal place, but *we may fill in any required number of decimal places with cyphers;* thus we may write the dividend 32,495·400, and then, moving the point three places to the right, we obtain the whole number 32,495,400.

Hence we divide 32,495,400 by 288.

EXAMPLE (3).—·00004 ÷ ·000000017.

To make the divisor a whole number, move the decimal point nine places to the right; the divisor becomes 17. Writing the dividend as ·000040000, and moving the decimal point nine places to the right, we obtain 40,000. Hence we must divide 40,000 by 17.

EXPLANATION.—To justify this rule we must first note that the decimal point is moved the same number of places to the right in both divisor and dividend; now, this is equivalent to *multiplying both divisor and dividend by the same quantity* (for instance, if we move the point two places to the right, we are multiplying both divisor and dividend by 100; cf. § 4); but if both are multiplied by the same quantity, *the number of times which the dividend contains the divisor will not be altered;* hence we get the correct quotient.

EXAMPLES.—V.

Evaluate—

1. ·003482 ÷ ·08; ·23225 ÷ ·005; ·0003456 ÷ ·9; 3249 ÷ ·05; 5 ÷ ·008.

2. 3420·1 ÷ ·023; 23 ÷ ·125; 2·242 ÷ 3·2; 135 ÷ 6·25; 112 ÷ ·00256; 23·4 ÷ 28·8.

3. Evaluate to seven significant figures: 222·28 ÷ ·23; ·0034 ÷ 1·26; ·0281 ÷ 5·1; 1·1248 ÷ ·0133; 2·36 ÷ 1·9; ·00004 ÷ ·0425.

9. Repeating decimals.—In some decimal fractions a figure or a group of figures is repeated without stopping; *e.g.* ·2355555 etc. . . . to infinity (where the 5 is supposed to be repeated "for ever"), and ·24625625625 etc. . . . to infinity (where the group 625 is supposed to be repeated for ever). Such decimals are called **repeating decimals, or recurring decimals**; if only one figure is repeated, we represent this by putting this figure down once only, with *a dot immediately above it;* if a group of figures is repeated, we represent this by putting the group down once only, with *a dot immediately above the first and last figures in the group.* Thus the above two repeating decimals are written $\cdot 23\dot{5}$, and $\cdot 24\dot{6}2\dot{5}$. Similarly $\cdot\dot{7}$ denotes $\cdot 7777$ etc. . . . to infinity; $23\cdot 3\dot{1}48\dot{7}$ denotes $23\cdot 31487148714 87$ etc. . . . to infinity.

The best way to *read* these decimals is to insert the word "*recurring*" or "*repeating*" *before* the figure or group of figures which is to be repeated; thus $\cdot 23\dot{4}5\dot{8}$ would be read as "point two three, recurring four five eight."

If all the figures after the decimal point are repeated (as in $32\cdot\dot{2}8\dot{5}$), it is called a "**pure** recurring decimal;" if some of the figures after the decimal point are not repeated (as in $32\cdot 2\dot{8}\dot{5}$), it is called a "**mixed** recurring decimal."

10. To reduce a pure recurring decimal to a vulgar fraction.

RULE.—**In the numerator put the figures after the decimal point; in the denominator put as many nines as there are figures after the decimal point. Reduce the fraction to its lowest terms.**

EXAMPLE (1).—$\cdot\dot{2}3\dot{4} = \frac{234}{999}$, which reduces by cancelling to $\frac{26}{111}$.

EXAMPLE (2).—$2\cdot\dot{4} = 2\frac{4}{9}$.

EXAMPLE (3).—$24\cdot\dot{0}3\dot{7} = 24\frac{037}{999} = 24\frac{37}{999}$, which reduces to $24\frac{1}{27}$.

EXPLANATION.—Taking Example (2), we wish to show that $\cdot\dot{4} = \frac{4}{9}$.

$$\text{Now } \cdot\dot{4} = \cdot 4444 \ldots \text{ to infinity.}$$
$$\therefore 10 \text{ times } \cdot\dot{4} = 4\cdot 444 \ldots \text{ to infinity (see § 4).}$$

Therefore, subtracting the upper line from the lower, we see that 9 times $\cdot\dot{4} = 4$; hence $\cdot\dot{4} = \frac{1}{9}$ of $4 = \frac{4}{9}$.

Again, taking Example (1)—

$$\cdot\dot{2}3\dot{4} = \cdot 234234234 \ldots \text{ to infinity.}$$
$$\therefore 1000 \text{ times } \cdot\dot{2}3\dot{4} = 234\cdot 234234 \ldots \text{ to infinity (see § 4).}$$

Therefore, subtracting the upper line from the lower, we see that 999 times $\cdot\dot{2}3\dot{4} = 234$.

$$\therefore \cdot\dot{2}3\dot{4} = \tfrac{1}{999} \text{ of } 234 = \tfrac{234}{999}$$

EXAMPLES.—VI.

Reduce to vulgar fractions in their lowest terms, or to mixed numbers—

1. $\cdot\dot{5}$. **2.** $\cdot\dot{7}\dot{2}$. **3.** $2\cdot\dot{8}\dot{1}$. **4.** $5\cdot\dot{1}4\dot{8}$. **5.** $12\cdot\dot{0}2\dot{7}$. **6.** $\cdot\dot{5}8\dot{5}$.
7. $1\cdot\dot{4}0\dot{5}$. **8.** $21\cdot\dot{0}01\dot{1}$. **9.** $\cdot\dot{9}2\dot{5}$. **10.** $\cdot\dot{1}4285\dot{7}$. **11.** $\cdot\dot{8}5714\dot{2}$.
12. $\dot{7}1428\dot{5}$. **13.** $\cdot\dot{3}8461\dot{5}$. **14.** $\cdot\dot{1}5384\dot{6}$. **15.** $\cdot\dot{2}3076\dot{9}$. **16.** $\cdot\dot{7}6923\dot{0}$.

11. To reduce mixed recurring decimals to vulgar fractions.

RULE.—**Put down all the figures after the decimal point; subtract from them those which do not recur; the result gives the numerator of the required fraction. In the denominator place as many nines as there are recurring decimals, followed by as many cyphers as there are non-recurring decimals.**

EXAMPLE (1).—$\cdot 23\dot{8}1\dot{1}$.

Following the above rule, we subtract 23 from 23811, which leaves 23788; this is the numerator of the fraction. Since the decimal contains three recurring figures and two non-recurring figures, the above rule gives 99900 as the denominator.

Hence $\qquad \cdot 23\dot{8}1\dot{1} = \frac{23788}{99900} = \frac{5947}{24975}$

EXAMPLE (2).—$23\cdot 328\dot{1}$.

Subtracting 328 from 3281, we have 2953. Also the decimal contains one recurring figure and three non-recurring figures. Hence, by the rule, $23\cdot 328\dot{1} = 23\frac{2953}{9000}$.

EXPLANATION.—Taking Example (1)—

$$\cdot 23\dot{8}1\dot{1} = \cdot 23811811811, \text{ etc. . . . to infinity;}$$
$$\therefore 100{,}000 \text{ times } \cdot 23\dot{8}1\dot{1} = 23811\cdot 811811, \text{ etc. . . . to infinity;}$$
$$\text{and } 100 \text{ times } \cdot 23\dot{8}1\dot{1} = \quad 23\cdot 811811811, \text{ etc. . . . to infinity.}$$

Thus subtracting the last line from the preceding, it follows that 99900 times $\cdot 23\dot{8}1\dot{1} = 23{,}788$.

$$\therefore \cdot 23\dot{8}1\dot{1} = \frac{1}{99900} \text{ of } 23{,}788 = \frac{23788}{99900}$$

By trying several cases for himself in a similar way, the student will soon see that the above rule will always hold; the important point in the proof is that the *first multiplication* must carry the decimal point to the *end* of the *first* group of the recurring figures, while the *second multiplication* must carry the decimal point to the *end* of the *non-recurring* figures.

EXAMPLES.—VII.

Reduce to the equivalent vulgar fractions in their lowest terms, or to mixed numbers—

1. $\cdot 1\dot{6}$. **2.** $\cdot 34\dot{2}\dot{1}$. **3.** $21\cdot 3\dot{9}\dot{6}$. **4.** $\cdot 3\dot{0}2\dot{7}$. **5.** $\cdot 84\dot{0}\dot{9}$.
6. $3\cdot 11\dot{2}\dot{7}$. **7.** $\cdot 03\dot{3}7\dot{8}$. **8.** $3\cdot 5743\dot{2}$. **9.** $\cdot 82\dot{2}60\dot{7}$. **10.** $\cdot 219\dot{0}59\dot{4}$.

12. To convert a vulgar fraction into the equivalent decimal fraction.

RULE.—**Divide the numerator by the denominator, by the method of decimal division.**

EXAMPLE (1).—*Reduce* $\frac{4}{5}$ *to a decimal.*

```
5)4·0
   ·8
```

EXAMPLE (2).—*Reduce* $\frac{7}{125}$ *to a decimal.*

```
125)7·00(·056
     625
    ----
     750
     750
```

EXAMPLE (3).—*Reduce* $\frac{4}{9}$ *to a decimal.*

```
9)4·000
  ·44444 etc. . . . to infinity.
```

Hence the equivalent decimal is $\cdot\dot{4}$.

EXPLANATION.—Taking Example (1), we know from vulgar fractions that $4 \div 5 = \frac{4}{5}$; but using the decimal rules, $4 \div 5 = \cdot 8$. Hence $\cdot 8$ must be equivalent to $\frac{4}{5}$. Similar reasoning applies in all cases.

EXAMPLES.—VIII.

Reduce to their equivalent decimal fractions—

1. $\frac{3}{8}$. **2.** $\frac{5}{16}$. **3.** $\frac{7}{32}$. **4.** $\frac{9}{64}$. **5.** $\frac{2}{5}$. **6.** $\frac{4}{25}$. **7.** $\frac{7}{125}$. **8.** $\frac{12}{625}$. **9.** $\frac{11}{16}$. **10.** $\frac{13}{25}$. **11.** $\frac{13}{400}$. **12.** $\frac{17}{250}$. **13.** $\frac{7}{320}$. **14.** $\frac{2}{3}$. **15.** $\frac{5}{9}$. **16.** $\frac{7}{9}$. **17.** $\frac{1}{6}$. **18.** $\frac{5}{6}$. **19.** $\frac{7}{12}$. **20.** $\frac{5}{12}$.

13. The following examples, which are harder cases of the preceding rule, require more careful study :—

EXAMPLE (1).—*Reduce $\frac{3}{11}$ to the equivalent decimal fraction.*

```
11)3·00
   ·272727 . . . to infinity.
```

$\therefore \frac{3}{11} = \cdot\dot{2}\dot{7}$.

EXAMPLE (2).—*Reduce $\frac{17}{22}$ to the equivalent decimal fraction.*

```
22)17·0(·77272 . . . to infinity.
   154
   ---
    160
    154
    ---
      60
      44
      --
      160
      154
      ---
        60
        44
        --
        160
      etc. . . .
```

Hence $\frac{17}{22} = \cdot 7\dot{7}\dot{2}$

It should be noticed in this example that the *third* remainder is the same as the *first* remainder (each is 16); it follows that the *fourth* step in the division is the same as the *second* (in each we divide 160 by 22, giving 7 in the quotient, and a remainder 6); also that the *fifth* step is the same as the *third* step (each gives 2 in the quotient, and a remainder 16). For similar reasons, the sixth and seventh steps will be the same as the second and third steps.

In general, it is obvious that if the same remainder occurs twice, *that portion of the division process which follows this remainder after it occurs the first time will be repeated after it occurs the second time;* and since this must again lead back to the same remainder, the *whole process* (including the figures in the quotient) *is repeated indefinitely.*

EXAMPLE (3).—*Reduce* $\frac{73}{148}$ *to its equivalent decimal fraction.*

```
148)73·0(·49324
    592
    ----
    1380
    1332
    ----
      480
      444
      ---
       360
       296
       ---
        640
        592
        ---
         48
```

Now, this last remainder, 48, is the same as the second remainder; hence from this point the process continually repeats itself, and the corresponding figures in the quotient, viz. 324, are continually repeated.

Hence $\frac{73}{148} = \cdot 49\dot{3}2\dot{4}$.

EXAMPLES.—IX.

Reduce to the equivalent decimal fractions—

1. $\frac{7}{18}$. **2.** $\frac{13}{36}$. **3.** $\frac{7}{44}$. **4.** $\frac{12}{55}$. **5.** $\frac{6}{275}$. **6.** $\frac{18}{37}$. **7.** $\frac{22}{111}$. **8.** $\frac{13}{148}$. **9.** $\frac{25}{303}$. **10.** $\frac{37}{606}$. **11.** $\frac{1231}{1818}$. **12.** $\frac{348}{2525}$. **13.** $\frac{125}{404}$. **14.** $\frac{3}{7}$. **15.** $\frac{5}{7}$. **16.** $\frac{2}{13}$. **17.** $\frac{5}{13}$. **18.** $\frac{12}{13}$.

14. When we wish to perform **addition, subtraction, multiplication, or division, with recurring decimals,** it is probably the simplest rule to reduce them to vulgar fractions, and then perform the required operations; if desired, the answer may be again converted into decimals.

EXAMPLE (1).—*Subtract* $\cdot 03\dot{3}7\dot{8}$ *from* $\cdot 11\dot{2}\dot{7}$.

Converting the decimals into vulgar fractions, we obtain $\frac{5}{148}$ and $\frac{31}{275}$.

$$\frac{31}{275} - \frac{5}{148} = \frac{4588 - 1375}{40700} = \frac{3213}{40700}$$

Reducing the result to a decimal, we obtain $\cdot 07\dot{8}9434\dot{8}$.

EXAMPLE (2).—*Divide* $\cdot 7\dot{8}\dot{4}$ *by* $1 \cdot 6\dot{8}\dot{1}$.

Converting into vulgar fractions, we obtain $\frac{259}{330}$ and $1\frac{15}{22}$.

$$\frac{259}{330} \div 1\frac{15}{22} = \frac{259}{330} \times \frac{22}{37} = \frac{7}{15}$$

Reducing the result to a decimal, we obtain $\cdot 4\dot{6}$.

There are other methods of working with recurring decimals; but in practical use they are of little or no importance.

EXAMPLES.—X.

Evaluate—

1. $\cdot 7\dot{6}\dot{3} + \cdot 5\dot{7}0\dot{2}$. **2.** $\cdot 7\dot{6}\dot{3} - \cdot 5\dot{7}0\dot{2}$. **3.** $\cdot 7\dot{8}\dot{4} \times \cdot\dot{5}9\dot{4}$.
4. $\cdot 2\dot{6} \times \cdot\dot{7}42\dot{5}$. **5.** $\cdot 6\dot{2} \div 1\cdot 02\dot{7}$. **6.** $\cdot\dot{3}2\dot{4} \div \cdot 3\dot{0}2\dot{7}$.

Evaluate correctly to seven significant figures—

7. $\cdot 4\dot{1}\dot{5} \times 2\cdot 21\dot{7}$. **8.** $\cdot 232 \div 3\cdot\dot{1}1\dot{7}$. **9.** $3 \div 2\cdot\dot{1}\dot{7}$.

15. In this connection, the following is an interesting theorem: No decimal fraction, **derived from a vulgar fraction,** can extend to infinity **without repeating.**

As an example, let us consider the fraction $\frac{5}{17}$. In the process of dividing the numerator by 17, we can only have 17 *different* remainders, viz. 0, 1, 2, 3, etc. . . . 14, 15, 16. If the remainder 0 occurs, the division terminates at this point; but if not, there are only 16 other *different* remainders which can occur; hence the seventeenth remainder *must* have occurred before; and a remainder *may have been repeated before this point.* But we have seen in § 13 that when a remainder is repeated the decimal recurs; thus the decimal may terminate or may repeat, but cannot go on for ever without repeating.

A decimal which terminates is called a **finite decimal.**

16. From the following considerations we can easily determine whether a given vulgar fraction will reduce to a finite or to a recurring decimal.

If we reduce any finite decimal to its equivalent vulgar fraction, the denominator is some "power of 10,"* such as 100, 10,000, etc. Consequently, when reduced to its prime factors, the *denominator contains only powers of* 2 *and* 5; although some of these may be cancelled in the process of reducing the fraction to its lowest terms, it will still hold that the denominator of the vulgar fraction in its lowest terms *will contain no factors other than powers of* 2 *or* 5. Conversely, **any fraction in its lowest terms, whose denominator contains other factors beside powers of 2 and 5, cannot reduce to a finite decimal.**

For example, in the case of $\frac{37}{250}$ the denominator $250 = 2 \times 5^3$; hence this will reduce to a finite decimal; in the case of $\frac{5}{52}$ the denominator $52 = 2^2 \times 13$, hence this will reduce to a repeating decimal.

17. The following list of results is, perhaps, worth remembering:—

$$\frac{1}{7} = \cdot\dot{1}4285\dot{7}$$
$$\frac{2}{7} = \cdot\dot{2}8571\dot{4}$$
$$\frac{3}{7} = \cdot\dot{4}2857\dot{1}$$
$$\frac{4}{7} = \cdot\dot{5}7142\dot{8}$$
$$\frac{5}{7} = \cdot\dot{7}1428\dot{5}$$
$$\frac{6}{7} = \cdot\dot{8}5714\dot{2}$$

It will be noticed that the same cycle of figures recurs in each decimal, but that they start at different points in the cycle.

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* See § 33.

EXAMPLES.—XI.

1. Add $\frac{3}{4}$ of ·235 to $\frac{5}{7}$ of ·5824.

2. From ·234 × 3·55 subtract ·128 ÷ 1·125.

3. Find the sum of $\frac{3}{5}$ of ·28, $\frac{7}{9}$ of ·342, $\frac{2}{7}$ of 14·581, $1\frac{2}{11}$ of 23·672.

4. Five experiments have been made to determine the exact weight of a certain piece of brass in ounces; the results were as follows: 2·003, 2·011, 1·997, 1·991, 2·001. Find the average of these results, correct to the third decimal place.

5. Find the value of $\frac{3}{17}$ of $\frac{2}{5}$ of $\cdot\dot{3}\dot{4}$.

6. Find correctly to four significant figures the value of $\frac{3}{11}$ of $1\frac{2}{7}$ of 72·3.

7. Find the average rainfall for the month of August for the last six years, if the observed rainfalls were 2·48, 2·10, 1·82, 1·91, 2·23, 2·05, giving the result correct to the second decimal place.

8. The length of a piece of copper wire is 24·23 inches, and its weight is 3·25 ozs. Find the average weight of the wire per inch, correct to four significant figures.

9. A clock loses on the average 2·0234 seconds per day; if it is set right on the 1st of January, at what date will it be 5 minutes slow?

10. Find the difference between the following quantities, giving the result correct to five significant figures: $\frac{2}{3}$ of $\frac{5}{7}$ of ·3481, and $\frac{19}{27}$ of ·023 × 51·21.

11. Evaluate 31·2 × 2·45 × 23·8 ÷ ·192.

12. A rectangular tank measures 12·3 inches in height, 32·2 inches in length, and 14·3 inches in breadth. If a cubic foot of water weighs 997 ozs., find the total weight (in pounds) of water when the tank is full, and the water pressure per sq. inch on the bottom of the tank, both correct to three significant figures.

CHAPTER II.

RATIO AND PROPORTION.

18. Ratio.—Ratio is the relation between two quantities of the same kind, in respect of their relative magnitude.

Consider the two quantities £6 and £10; the first quantity is obviously $\frac{3}{5}$ of the second quantity; and this is one method of stating their relative magnitude. In the language of "ratio," the same idea is expressed by saying that these two sums of money are in the "**ratio of 3 to 5;**" and this ratio is expressed by the notation **3 : 5.**

It should now be clear that, when we say that two quantities are in the ratio 7 : 11, we mean that the first quantity is $\frac{7}{11}$ of the second quantity, or (what amounts to the same thing) that the second quantity is $\frac{11}{7}$ of the first.

The two numbers in a ratio are called the "**terms**" of the ratio. The first is called the "**antecedent**" (meaning "*that which goes before*"); the second is called the "**consequent**" (meaning "*that which follows after*").

19. It is clear from the preceding paragraph that, in order **to find the ratio of two quantities,** we need only find what fraction the first quantity is of the second.

EXAMPLE.—*Find the ratio of* 2 *feet* 3 *inches to* 6 *feet.*

$$\left.\begin{array}{ll}2\text{ feet }3\text{ inches} & = 27\text{ inches}\\ 6\text{ feet} & = 72\text{ inches}\end{array}\right\}\ \frac{27}{72}=\frac{3}{8}$$

Thus the first quantity is $\frac{3}{8}$ of the second; or the quantities are in the ratio of **3 : 8.**

Again, since a ratio is the equivalent of a vulgar fraction; and since we do not alter the value of a vulgar fraction by multiplying or dividing both numerator and denominator by the same quantity; the same rule will obviously apply to the terms of a ratio. For example, the ratio 1200 : 1800 can be "**reduced**" to the ratio **2 : 3** by dividing both antecedent and consequent by 600.

If the antecedent is greater than the consequent, it is termed a **ratio of greater inequality**; if the opposite, it is termed a **ratio of less inequality**; if the two terms are equal, we have a ratio of **equality.**

20. The following examples will show the principles involved:—

EXAMPLE (1).—*The ratio of a certain sum of money to* 16*s.* 8*d. is* 17 : 25; *find that sum.*

The required sum is $\frac{17}{25}$ of 16*s.* 8*d.* = $\frac{17}{25}$ of 200*d.* = 136*d.* = 11*s.* 4*d.*

EXAMPLE (2).—*If the ratio of* A *to* B *is* 3 : 8, *and the ratio of* B *to* C *is* 28 : 27, *find the ratio of* A *to* C.

We are given that A is $\frac{3}{8}$ of B, and that B is $\frac{28}{27}$ of C; hence A is $\frac{3}{8}$ of $\frac{28}{27}$ of C, *i.e.* A is $\frac{7}{18}$ of C.

∴ the ratio of A to C is 7 : 18

EXAMPLE (3).—*Reduce the ratio* 16 : 27 *to an equivalent ratio whose antecedent is* 1.

We must divide both antecedent and consequent by 16. Dividing 27 by 16, we obtain 1·6875. Hence the equivalent ratio is 1 : 1·6875.

Sometimes the ratio of one quantity to another is expressed as a decimal; the decimal being the equivalent of the vulgar fraction which corresponds to the ratio.

EXAMPLE (4).—*Express as a decimal the ratio of* 5¾ *tons to* 12 *tons* 10 *cwt.*

5¾ tons : 12 tons 10 cwt. = 115 cwt. : 250 cwt. = 23 : 50. The equivalent vulgar fraction is $\frac{23}{50}$, which reduces to ·46.

NOTE.—This is obviously the same as the question, "What decimal of 12 tons 10 cwt. is 5¾ tons?"

EXAMPLES.—XII.

Find the ratio of the following pairs of quantities:—

1. 3*s.* 4*d.*, 10*s.* 6*d.* **2.** $\frac{1}{10}$ mile, 10 poles. **3.** 1000 ozs., ½ cwt.
4. 3000 sq. inches, 5 sq. feet. **5.** £100, 2000 francs [£1 = 25 francs].
6. 100 guineas, 100,000 farthings.
7. What quantity is to 17*s.* 6*d.* in the ratio of 24 : 35?
8. The ratio of the circumference of a circle to its diameter is 355 : 113. Find the diameter of a circle whose circumference is 17 feet 9 inches.
9. The ratio of a side of a square to its diagonal is 500 : 707. Find, correct to four significant figures, the side of a square whose diagonal is 17 inches.
10. If the ratio of A's salary to B's is 4 : 5, and the ratio of B's salary to C's is 2 : 3, find the ratio of C's salary to A's. Also find A's salary, given that C's is £375 per annum.
11. Reduce the following ratios to equivalent ratios whose antecedent is unity, working to three significant figures: (i.) 20 : 35; (ii.) 16 : 19; (iii.) 8 : 5; (iv.) 7 : 3; (v.) 330 : 729.
12. Express the following ratios as decimal fractions, correct to four significant figures: (i.) £10 : 21 guineas; (ii.) 3 acres : 10,000 sq. yards; (iii.) 2700 cub. inches : 2·5 cub. feet; (iv.) ·000125 feet : ·00234 inches.

21. Per-centages. The ratio of two quantities is often expressed as a **per-centage.** The word "**cent**" means "a hundred;" thus to say that a regiment has lost **3 per cent.** of its men means that 3 *men out of every* 100 *men* were lost; similarly, if a man's yearly income is **11 per cent.** of his capital, he is getting a yearly income of £11 *for every* £100 *of capital* which he possesses.

RULE.—**To convert a per-centage into a fraction, place the percentage in the numerator, and 100 in the denominator.**

EXAMPLE (1).—*Express* 15 *p.c. as a fraction.*

$$15 \text{ p.c.} = \frac{15}{100} = \frac{3}{20}$$

EXPLANATION.—To take 15 in every 100 is obviously to take $\frac{15}{100}$ of the whole quantity.

EXAMPLE (2).—*Find* $\frac{3}{8}$ *p.c. of* £120.

$$\tfrac{3}{8}\text{ p.c.} = \frac{\frac{3}{8}}{100} = \tfrac{3}{800}$$

$$\tfrac{3}{800}\text{ of } £120 = 9\text{ shillings}$$

EXPLANATION.—If in every £100 we are to take £$\frac{3}{8}$, in every £1 we must take £$\frac{3}{8} \div 100$, *i.e.* £$\frac{3}{800}$; hence we must take $\frac{3}{800}$ of the whole sum.

EXAMPLES.—XIII.

1. Reduce the following per-centages to the equivalent fractions:—
(i.) 25 p.c.; (ii.) 35 p.c.; (iii.) $12\frac{1}{2}$ p.c.; (iv.) $33\frac{1}{3}$ p.c.; (v.) $16\frac{2}{3}$ p.c.; (vi.) 7·5 p.c.; (vii.) 1·25 p.c.; (viii.) $\frac{1}{3}$ p.c.; (ix.) $\frac{2}{5}$ p.c.; (x.) $112\frac{1}{2}$ p.c.; (xi.) 137·5 p.c.; (xii.) ·375 p.c.

2. Evaluate the following: (i.) $22\frac{1}{2}$ p.c. of £8000; (ii.) 17 p.c. of 450 acres; (iii.) 28 p.c. of 250 gallons; (iv.) ·2 p.c. of £500; (v.) $\frac{2}{7}$ p.c. of £18 4*s.*; (vi.) 92 p.c. of 3425 children.

3. 7·5 p.c. of the population of a village die during an epidemic. If the population was 680 before the epidemic, what was it afterwards?

4. If the wheat-harvest at a farm is 12·5 p.c. better in a certain year than in the preceding year; and if in the preceding year the harvest was 3248 bushels, find the number of bushels in the second year.

22. To express a fraction as a per-centage.
RULE.—**Multiply the fraction by 100.**

EXAMPLE (1).—*Convert* $\frac{3}{16}$ *into a per-centage.*
$\frac{3}{16} \times 100 = \frac{75}{4} = 18·75$; hence $\frac{3}{16}$ is equivalent to 18·75 p.c.

EXPLANATION.—If we are to take $\frac{3}{16}$ of a number—from every 100 we must take $\frac{3}{16}$ of 100, *i.e.* 18·75; hence we are taking 18·75 p.c.

EXAMPLE (2).—*How much per cent. is* 24 *of* 192?
First find what fraction 24 is of 192; result, $\frac{24}{192} = \frac{1}{8}$.
Reducing $\frac{1}{8}$ to a per-centage, we obtain $\frac{1}{8}$ of 100, *i.e.* 12·5 p.c.

EXAMPLE (3).—4240 *gallons of water were put into a tank, and after a week it contained only* 4224 *gallons, owing to evaporation. Find, correct to three significant figures, the per-centage lost by evaporation per day.*
The loss in 7 days was 16 gallons.

$$\therefore \text{ the loss in 1 day was } \tfrac{16}{7} \text{ gallons}$$

$$\therefore \text{ the fractional loss in 1 day} = \frac{\frac{16}{7}}{4240} = \tfrac{1}{1855}$$

$$\therefore \text{ the per-centage loss in 1 day} = \tfrac{1}{1855} \times 100 = \tfrac{20}{371} = ·0539 \text{ p.c.}$$

EXAMPLES.—XIV.

1. What per-centage is 45 in 120?
2. What per-centage is 1·75 in 5?
3. How much per cent. is 23 in 345? (Correct to four significant figures.)

4. A gold coin when fresh from the mint weighs exactly ·24 oz. After being in circulation for 3 years, it is found to weigh ·234 oz.: find the per-centage loss of the metal.

5. An electrical machine was bought for £1050; it costs on the average £13 a month to keep it working: what per-centage is the annual working expense of the original outlay?

6. In manufacturing a machine the ratio of the cost of the material to the cost of the workmanship is 17 : 29; and the manufacturer's profit is equal to the cost of the workmanship: what per-centage is the profit on the total outlay? (Give the answer correct to four significant figures.)

23. The following type of problem should be carefully noted:—

EXAMPLE (1).—*£28 is 37·5 p.c. of a certain sum. Find that sum.*

$$37{\cdot}5 \text{ p.c.} = \frac{37{\cdot}5}{100} = \frac{3}{8}.$$

Thus £28 is $\frac{3}{8}$ of the required sum.

∴ The required sum is $\frac{8}{3}$ of £28 = £74 13*s.* 4*d.*

EXAMPLE (2).—*·023 p.c. by weight of a certain gold-bearing quartz is pure gold. How many tons of quartz must be crushed to give 2 lbs. of gold?*

$${\cdot}023 \text{ p.c.} = \frac{{\cdot}023}{100} = \frac{23}{100000}$$

Thus the weight of gold produced is $\frac{23}{100000}$ of the weight of quartz crushed. Therefore the weight of quartz crushed is $\frac{100000}{23}$ of the weight of gold produced.

Hence to produce 2 lbs. of gold requires $\frac{100000}{23}$ of 2 lbs. of quartz = 3·88 tons.

EXAMPLES.—XV.

1. £45 is 15 p.c. of what sum?

2. On what weight does 18·9 lbs. amount to ·27 p.c.?

3. 2·3 pints has leaked out of a cask of spirit, and this is known to be 8·05 p.c. of the quantity originally in the cask: find the quantity left in the cask.

4. 3·2 p.c. of a regiment has been lost in a campaign, and the survivors number 1452: find the original strength of the regiment.

5. The composition of a coarse gunpowder by weight is as follows: 74·8 p.c. of nitre, 10·3 p.c. of sulphur, 13·9 p.c. of charcoal, and 1 p.c. of water: find the actual weight of each ingredient in 3·75 ozs. of powder.

6. If a fine-grained gunpowder contains 73·6 p.c. (by weight) of nitre, 10·3 p.c. of sulphur, 14·6 p.c. of charcoal, and 1·5 p.c. of water: find what weight of this gunpowder can be made from 4·38 lbs. of charcoal; and what weight of sulphur is used in the process.

24. Proportion.—An equality of ratios is called a "proportion." For example, we know that the ratio 12 : 8 is equal to the ratio 9 : 6, for each reduces to the ratio 3 : 2; hence we may write 12 : 8 = 9 : 6, and this statement is called a **proportion.** The same statement is often expressed by saying that **the four numbers, 12, 8, 9, 6 are in proportion.** The first and fourth numbers in a proportion are called the **extremes,** and the other two are called the **means.**

THEOREM.—**In any proportion the product of the extremes is equal to the product of the means.**

To prove this, suppose that (for the sake of brevity) we represent the four quantities in proportion by the letters a, b, c, and d, respectively; then the ratio $a : b$ is equal to the ratio $c : d$. But the ratio $a : b$ is not altered in value if we multiply both terms by d (see § 19), which gives the ratio $a \times d : b \times d$. Similarly, the ratio $c : d$ is not altered in value if we multiply both terms by b, which gives the ratio $b \times c : b \times d$.

$$\text{Hence the ratios } \left.\begin{array}{r} a \times d : b \times d \\ \text{and } b \times c : b \times d \end{array}\right\} \text{ are equal;}$$

but their *consequents* are the same; thus *they cannot be equal ratios unless their antecedents are also the same;*

$$\therefore\ a \times d = b \times c$$

i.e. the **product of the extremes is equal to the product of the means.**

25. To find the fourth proportional to three given quantities; *i.e.* given the first three terms in a proportion, to find the fourth.

RULE.—**Multiply the second and third terms, and divide the product by the first term.**

EXAMPLE.—*Find the fourth proportional to* 9, 21, *and* 12.

$$21 \times 12 \div 9 = 28$$

EXPLANATION.—We have $9 : 21 = 12 :$ required number.

Thus (by § 24) $9 \times$ required number $= 21 \times 12$

$\therefore$ the required number $= 21 \times 12 \div 9$

26. If a, b, and c are quantities such that $a : b = b : c$, then c is called the **third proportional** to a and b, also b is called the **mean proportional** to a and c.

To find the third proportional to two given numbers.

RULE.—**Multiply the second number by itself, and divide by the first.**

EXAMPLE.—*Find the third proportional to* 9 *and* 12.

$$12 \times 12 \div 9 = 16$$

EXPLANATION.—If we require the third proportional to 9 and 12, then $9 : 12 = 12 :$ required number.

$\therefore$ (by § 24) $9 \times$ required number $= 12 \times 12$

$\therefore$ the required number $= 12 \times 12 \div 9$

To find the mean proportional to two given numbers.

RULE.—**Multiply the numbers, and take the square root of their product.** (For definition of square root, see § 36.)

EXAMPLE.—*Find the mean proportional to* 18 *and* 50.

$$18 \times 50 = 900. \quad \sqrt{900} = 30$$

EXPLANATION.—Since the required number is the mean proportional,

$\therefore$ 18 : required number = required number : 50

hence (by § 24) the square of the required number $= 18 \times 50 = 900$

$\therefore$ the required number $= \sqrt{900} = 30$

EXAMPLES.—XVI.

Find the fourth proportional to—

1. 6, 8, 21. **2.** 10, 14, 15. **3.** 12, 20, 21. **4.** 7, 20, 35.
5. 8, 50, 12. **6.** 18, 33, 30.

Find the third proportional to—

7. 4, 6. **8.** 9, 12. **9.** 4, 12. **10.** 3, 15. **11.** 9, 21. **12.** 144, 36.

Find the mean proportional to—

13. 4, 25. **14.** 8, 18. **15.** 4, 36. **16.** 48, 3. **17.** 32, 2. **18.** 18, 2.

EXAMPLES.—XVII.

1. Find the ratio of the average rates of two trains, if one travels 420 miles in 8 hours, and the other 450 miles in 12 hours.

2. 27 cub. inches of copper weigh 125 ozs., and 72 cub. inches of iron weigh 312·5 ozs.: find the ratio of the weights of equal volumes of the two metals.

3. Gold, silver, and copper are melted together to form a coin in weights proportional to the numbers 32, 5, 3: what per-centage by weight is the copper of the whole?

4. Three partners A, B, C, are working together in a business; the ratio of A's capital to B's is 3 : 2, and the ratio of B's capital to C's is 6 : 5: what per-centage of the profits should be given to C?

5. A sample of brandy on analysis gave the following results: pure spirit, 2·24 ozs.; pure water, 3·27 ozs.; other ingredients, 0·21 oz.: reduce the result to per-centages correct to the second decimal place.

6. The following is an analysis of sea-water: pure water, 96·47 p.c.; pure sodium chloride (common salt), 2·71 p.c.; the residue consisting of various other mineral substances: how many tons of sea-water must be taken to give 1 cwt. of pure sodium chloride?

7. Using the data of the preceding question, find the ratio of two quantities of sea-water such that the weight of pure water in the first is equal to the weight of pure sodium chloride in the second; express the ratio as a decimal correct to three significant figures.

8. Given the following analysis of air by volume: nitrogen, 78 p.c.; oxygen, 20·6 p.c.; aqueous vapour, 1·4 p.c.;—if all the aqueous vapour were removed, what per-centage of the residue would be oxygen?

9. Using the data of the preceding question, if half the aqueous vapour and half the nitrogen were removed, what per-centage of the residue would be oxygen?

CHAPTER III.

CONTRACTED METHODS OF MULTIPLICATION AND DIVISION.

27. On degree of accuracy.—In practical work it is impossible to obtain perfect accuracy. We can *calculate* as accurately as we wish; but the difficulty is to *measure accurately* the quantities on which our calculations are based. For instance, if we are measuring lengths with a foot-rule, the length of the rule is probably a few thousandths of an inch more or less than a foot, and it is also extremely improbable that all the divisions on the rule are of exactly equal length; moreover, even if we could get a perfect foot-rule, and if the eye could estimate exactly what length on the rule corresponded to the length we wished to measure, it is extremely improbable that this length would correspond to an exact number of divisions on the rule.

However, by various complicated methods and instruments, it is possible to measure much more accurately than would be naturally thought possible. But, however exact the method, we have to be satisfied if we are sure that the error is *very small compared with the quantity measured.* It is important to notice that the actual magnitude of the error is not important, but the *ratio of the error to the quantity measured.* For example, in measuring the distance of the centre of the earth from the centre of the moon, it is impossible to be certain that we are not one or two miles out; but this error is of no consequence in a length which is about 240,000 miles. In measuring a rod which is only a few inches long, we can make certain of being correct to within a thousandth of an inch; while in measuring the wave-lengths of different coloured lights (which are less than $\frac{1}{10000}$ of an inch), we can ensure accuracy to within $\frac{1}{1000000000}$ of an inch.

28. Now, the degree of accuracy is best stated by the number of **significant figures** known to be correct.

We have already used the term "significant figures," but its meaning has not yet been fully explained.

It must be noticed that by the **first significant figure** we mean **the first figure on the left which is not a cypher**; thus in each of the quantities 32·58, ·0003258, the first significant figure is 3. It is obvious that the first significant figure is the figure which has the greatest value in the quantity.

The other significant figures are numbered in order from the first significant figure, and count as significant figures, *whether they are* *cyphers or not;* thus in the quantity ·000320047, the first three significant figures are 320.

Now, we can show that if a result is correct to **four** significant figures, the ratio of the error to the result is less than **1 : 1000**. Suppose, for example, the result 423457·8 is known to be correct to four significant figures; then the figures 4234 are known to be correct, and the figures 57·8 are doubtful; it follows that the error must be *less* than 100; while the result itself is *greater* than 100,000. Hence the ratio of error to result is less than 100 : 100,000, *i.e.* less than 1 : 1000.

Again, suppose the result ·0034572 is known to be correct to four significant figures. Then the figures ·003457 are correct; thus the error must be *less* than ·000001, while the result itself is *greater* than ·001; it follows that the ratio of error to result is less than ·000001 : ·001, *i.e.* less than 1 : 1000.

EXAMPLES.—XVIII.

1. Show that if the following results are correct to three significant figures the ratio of error to result is in each case less than 1 : 100:—324,287,261·3, 324,287·2613, ·000428761521.

2. In the preceding question, show that if the results are correct to seven significant figures, the ratio of error to result is in each case less than 1 : 1,000,000.

29. It should now be obvious that it is never of any practical use to work out an arithmetical result correct to very many significant figures. The errors introduced in the preliminary measurements will render unreliable all but the first few significant figures of the result.

If we are working with exceedingly accurate instruments, and require the greatest accuracy in our result, it is never reliable beyond the seventh significant figure; and in most cases it is only reliable to the fifth. For ordinary practical work, accuracy to four or even three significant figures is all that is required.

It is therefore very useful to have shortened methods of multiplication and division, which give us only the first few significant figures in the answer.

30. To multiply two quantities correctly to a given number of significant figures.

RULE.—**To multiply two quantities correctly to four significant figures.**

(a) **Multiply the first *five* significant figures in the upper line by the *first* significant figure in the lower line** (allowing for any quantity which would naturally be "carried" from the rest of the upper line). **Multiply the first *four* significant figures of the upper line by the *second* significant figure in the lower line,** the first *three* in the upper line by the *third* in the lower line, and so on (allowing in each case for any quantity which would naturally be "carried" from the rest of the upper line).

(b) These different lines of multiplication must be placed with their **right-hand figures in continuous column,** and then added.

(c) The **position of the decimal point** in the result is the same as its position in any of the lines of multiplication, which can easily be determined by the principles of Chapter I.

(d) The **last figure in the result is doubtful**, and must be rejected.

The application of the preceding rules will only become clear if the following examples are carefully studied :—

EXAMPLE (1).—30·2341 × 31·62 *correct to four significant figures.*

```
 30·2341
 31·62
 ———————
907·02
 30·23
 18·13
     60
 ———————
955·98
```

(a) We first multiply 30234 by 3; then we multiply 3023 by 1; then 302 by 6 (putting the product as 18·13, instead of 18·12, because we should have to "carry 1" from 6 times 3 had we multiplied the fourth significant figure by 6); then 30 by 2.

(b) These lines of multiplication are placed so that the figures on the right form a continuous column.

(c) The 2 at the end of the first line of multiplication was obtained by multiplying 3 in the multiplier by 4 in the multiplicand; now, these represent 3 tens and 4 thousandths respectively; but $10 \times \frac{1}{1000} = \frac{1}{100}$; thus 3 tens × 4 thousandths give 12 hundredths, which is 1-tenth and **2 hundredths**; consequently the **2** must be in the **second decimal place**. This determines the position of the decimal point in the first line.

We shall find that if our rules are correctly carried out, the position of the decimal point is the same in each of the other lines of multiplication; for instance, in the third line of multiplication we start by multiplying 6 in the multiplier by 2 in the multiplicand; each of these represents tenths; hence their product represents hundredths, and the figure obtained must therefore be in the second decimal place.

(d) There are two reasons which make the last figure in the result doubtful. Firstly, the figures in the column above it may not be quite correct, although we allowed for "carrying" in the figures which we did not multiply. For instance, if we multiply the whole multiplicand by 6, we obtain 1814046; thus our third line of multiplication ought to be 1814, not 1813. Secondly, in the full process of multiplication there would be other columns of figures to the right of those which we have here; and in the addition there would probably be some carrying from these columns which would affect the result. It is usual to allow roughly for these sources of error by carrying one too many *on to our right-hand column* whenever we are carrying from a product which ends in 5 or a larger figure.

Thus, in the third line of our multiplication we start by multiplying the 2 by 6, to which we add the figure to be carried from 6 times 3, and since this is 18, we carry 2 instead of 1. On the same principle, if we are carrying from 6 times 7, we should carry 4; but from 6 times 8 we should carry 5; and from 3 times 3 we should carry 1. But even this correction *does not protect the last figure from error*, but is rather a safeguard for the last figure but one. Thus we regard the result as correct only to four significant figures, which is the number required.

NOTE 1.—If we are using the above result, we should quote it as 956·0, because the figure that we are rejecting is greater than 5, and we therefore add 1 to the last figure retained.

NOTE 2.—Sometimes this law gives six significant figures in the result, in which case the first five are reliable.

NOTE 3.—We can, of course, apply the same method to finding a product correct to any number of significant figures. If we want to work correctly to seven significant figures, we multiply the first significant figure in the multiplier into the first eight significant figures in the multiplicand.

EXAMPLE (2).—·003456 × ·02804 *correct to four significant figures.*

(*a*) The first five significant figures in the multiplicand are 34560; these we multiply by 2; the first four are multiplied by 8, etc.

(*c*) The figures in the right-hand column were obtained by multiplying

2nd decimal place by 7th, in the first line;
3rd ,, ,, 6th, ,, second line;
5th ,, ,, 4th, ,, fourth line;

hence this column represents the ninth decimal place, which shows that we must prefix four cyphers and the decimal point to the result.

```
    ·0034560
    ·02804
  ----------
  ·000069120
       27648
           0
         138
  ----------
  ·000096906
```

EXAMPLE (3).—34,203·85 × 2,681·4 *correct to five significant figures.*

(*a*) We multiply 342038 by 2; 34203 by 6; etc.

(*c*) The figures in the right-hand column are obtained by multiplying

thousands by tenths in the first line;
hundreds by units ,, second line;
etc.

Hence the figures in this column represent hundreds, and we therefore affix two cyphers to the result.

(*d*) The result would be quoted as 91,714,000.

```
 34,203·85
  2,681·4
 ---------
 68407 700
 20522 3
  2736 2
    34 2
    13 7
 ---------
 91714 00
```

EXAMPLES.—XIX.

1. 23·2805 × ·2345	correct to	4	significant	figures.
2. ·030421 × ·827	,,	5	,,	,,
3. 3·682145 × 3248	,,	4	,,	,,
4. 20,045 × 3002·81	,,	3	,,	,,
5. ·0002358 × ·00387	,,	4	,,	,,
6. ·027835 × 258	,,	5	,,	,,
7. 2005·381 × 398·25	,,	5	,,	,,
8. 2,035,602 × ·0004873	,,	4	,,	,,
9. 243,283·7 × 237·268	,,	4	,,	,,
10. 34·26847 × 2·289	,,	3	,,	,,
11. 2048 × 5027	,,	5	,,	,,
12. 4821·3 × 294·8	,,	3	,,	,,

31. To divide correctly to a given number of significant figures.

RULE.—**To divide correctly to five significant figures:**

(a) Move the decimal point in the divisor so that it comes after the **sixth** significant figure ; **move the decimal point in the dividend** the same number of places in the same direction ; reject all figures in the divisor which now come after the decimal point.

(b) After each step in the division, **instead of bringing down another figure from the dividend, reject another figure from the divisor.**

(c) In each multiplication **take account of quantities which would naturally be "carried"** from the rejected figures in the divisor.

(d) The figure of the quotient which is given by the **first step of the division** must be put in the **same** position with regard to the decimal point as the **last figure** of the dividend which is used in the **first step** of the division. (This is, of course, the same rule as is given in § 6.)

EXAMPLE (1).—32·258571 ÷ 2873·28384 *correct to five significant figures.*

(a) To place the decimal point in the divisor after the sixth significant figure, we must move it two places to the right, and must therefore move it two places to the right in the dividend. The figures 384 in the divisor are rejected.

```
2,8,7,3,2,8·384)3225·8571(·0112271
                2873 28
                -------
                 352 57
                 287 33
                 ------
                  65 24
                  57 46
                  -----
                   7 78
                   5 75
                   ----
                   2 03
                   2 01
                   ----
                      2
                      2
```

(b) In the first step we divide by 287,328 ; in the second step we divide by 28,732 ; in the third step we divide by 2873 ; etc.

(d) The *last figure* of the dividend which is used in the *first step* of the division is in the *second* decimal place ; hence the quotient 1 is in the *second* decimal place.

Note that it would be more correct to count the last figure in the quotient as 0 than 1, for in accordance with our rule for carrying we should carry 1 from the rejected 8 in the divisor ; hence this would make the lowest figure 3 instead of 2. However, in any case the last figure in the quotient is unreliable, and we have the five significant figures without it.

EXPLANATION.—If the above process be compared with the full division process below, it will be obvious that the rule arranges for leaving out that portion of the working (to the right of the vertical line) which does not affect the first five significant figures in the quotient.

```
287328384)322585|7·10(·01122707 . . .
          287328|384
          ------|---
           35257|3260
           28732|8384
           -----|----
            6524|48760
            5746|56768
            ----|-----
             777|919920
             574|656768
             ---|------
             203|2631520
             201|1298688
             ---|-------
               2|133283200
               2|011298688
```

EXAMPLE (2).—32,148,559,142 ÷ 8471 *correct to four significant figures.*

(a) The divisor is 8471·00 . . . ; placing the decimal point after the fifth significant figure, we obtain 84,710·0 . . . ; the corresponding change in the dividend gives 321,485,591,420.

```
8,4,7,1,0)321485591420(3795\100
          254130
          ------
           67355
           59297
           -----
            8058
            7624
            ----
             434
             424
             ---
              10
               8
              --
               2
```

(b) In the first step we divide by 84,710; in the second by 8471; etc.

(d) The last figure of the dividend used in the first step of the division is in the "millions'" place; hence the 3 in the quotient represents millions.

The result would be quoted as 3,795,000.

A good arithmetician will save a little time and space by doing the multiplication and subtraction in one step and writing down only the remainders; the last example would then stand as follows:—

```
8,4,7,1,0)321485591420(3795\100
           67355
            8058
             434
              10
               2
```

The first step in the division is then worked thus—

3 times 0 are 0, 0 from 5 leaves 5;

3 times 1 are 3, 3 from 8 leaves 5;

3 times 7 are 21, 1 from 4 leaves 3, carry 2;

3 times 4 with 2 carried gives 14, 4 from 1 leaves 7, carry 1 from the multiplication, and 1 from the subtraction;

3 times 8 with 2 carried gives 26, 26 from 32 leaves 6.

EXAMPLES.—XX.

1. 83·234 ÷ 721·4 correct to 3 significant figures.
2. ·0093285 ÷ ·05873 ,, 3 ,, ,,
3. 9,283,756 ÷ 286·3 ,, 3 ,, ,,
4. 385,241 ÷ ·0287 ,, 4 ,, ,,
5. 73·2874 ÷ ·03875 ,, 5 ,, ,,
6. 2·8314 ÷ 3·287 ,, 6 ,, ,,
7. 20,000,000 ÷ 176·5 ,, 5 ,, ,,
8. 11,111·111 ÷ 6·217 ,, 4 ,, ,,
9. ·314159 ÷ ·4771213 ,, 5 ,, ,,
10. ·4771213 ÷ 31·4159 ,, 5 ,, ,,
11. 11,000,000 ÷ ·002876 ,, 4 ,, ,,
12. ·0003872 ÷ 200,597·3 ,, 4 ,, ,,

32. Many text-books on Arithmetic give rules for multiplying or dividing correct to a *given number of decimal places.* But these rules are rather of theoretical interest than of practical value, and we do not advise the student to trouble about them.

If we are finding the continued product of several quantities to a given number of significant figures by contracted methods, there is some fear that the error may accumulate in the repeated processes and ultimately affect the significant figures required correct. In this case it is best not to reject the last figure in the intermediate results, for we are certainly more correct in retaining it than in omitting it entirely; also if there are more than three quantities to be multiplied together, it will be better to work for one extra significant figure as a safeguard against accumulated error. The same remarks, of course, apply to the division process.

EXAMPLES.—XXI.

1. 23·28 × 59·27 ÷ 28·361 correct to 4 significant figures.
2. 37·23 × 2872 × 3005 ,, 4 ,, ,,
3. 823·268 × ·2043 ÷ ·00357 ,, 4 ,, ,,
4. ·02934 × ·06821 ÷ ·23456 ,, 5 ,, ,,
5. 20,555,000 × ·7258 ÷ 62·34 ,, 5 ,, ,,
6. 3·14159 × ·4771213 × 1·414213 × 1·73205 correct to 4 significant figures.
7. 2·236 × 2·449 × 23,205,871 correct to 5 significant figures.
8. Divide the product of 3·14159 and ·4771213 by the product of 1·414213 and 1·73205 correct to 5 decimal places.

CHAPTER IV.

INVOLUTION AND EVOLUTION.

33. Involution is the process of multiplying a quantity by itself any given number of times.

The result of such a multiplication is called a "**power**" of the quantity.

The **first power** of a number is the number itself;						
„ **second**	„	„	„	multiplied by itself **once**;		
„ **third**	„	„	„	„	„	**twice**;
„ **fourth**	„	„	„	„		**three times**;
etc.						etc.

The **second power** of a number is more usually called the "**square**" of the number.

The **third power** of a number is more usually called the "**cube**" of the number.

There are no special names for the higher powers of a number.

Thus the first power of 3 is 3;
the square or (second power) of 3 is 3×3, *i.e.* 9;
the cube (or third power) of 3 is $3 \times 3 \times 3$, *i.e.* 27;
the fourth power of 3 is $3 \times 3 \times 3 \times 3$, *i.e.* 81; etc.

Any power of a quantity is usually represented by writing the corresponding figure (written small) above and to the right of the quantity; thus $\mathbf{5^3}$ denotes the **third** power (or cube) of 5, which is $5 \times 5 \times 5$, or 125; $\mathbf{3^7}$ denotes the **seventh** power of 3, which is $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$, or 2187. These figures representing powers are spoken of as the "**indices**" (singular, "**index**") of those powers.

To find a power of any quantity is obviously a matter of multiplication.

34. There is one case in involution in which an easily remembered rule will often save trouble.

RULE.—**To square a number which ends in 5: reject the 5; multiply what is left by 1 more than itself; write 25 at the end of the product.**

EXAMPLES.—*Square* 65.

$6 \times 7 = 42$ Result, 4225

Square 195.

$19 \times 20 = 380$ Result, 38,025

EXPLANATION.—(The student who is not acquainted with Algebra must postpone this explanation till after he has read Chapter VIII.)

Let a represent the number left when the 5 is rejected ; then the value of the number is $10a + 5$.

[For instance, in the case of 195, a would be 19, and the value of the number is obviously $10 \times 19 + 5$.]

Thus the square of the number $= (10a + 5)^2 = 100a^2 + 100a + 25 = 100a(a + 1) + 25$. This gives the rule: "Multiply a by $a + 1$, multiply the result by 100, and add 25 ;" which is obviously equivalent to the rule given above.

There is also a rule for squaring a number which ends in 25, but its practical value is doubtful.

RULE.—**Multiply together the two numbers formed by removing (i.) the 2, (ii.) the 25; write 625 at the end of the product.**

EXAMPLES.—*Square* 725.

$$\begin{array}{r} 75 \\ 7 \\ \hline 525 \end{array} \qquad \text{Result, } 525{,}625$$

Square 8425.

$$\begin{array}{r} 845 \\ 84 \\ \hline 3380 \\ 6760 \\ \hline 70980 \end{array} \qquad \text{Result, } 7{,}0980{,}625$$

EXPLANATION.—If a is the number left when the 25 is rejected, the value of the number is $100a + 25$. Hence its square $= (100a + 25)^2 = 10000a^2 + 5000a + 625 = 1000a(10a + 5) + 625$; from which we derive the rule : "Multiply $10a + 5$ by a, and write 625 after the result ;" which is obviously equivalent to the above rule.

35. The following points are important :—

The fourth power of a number is the square of its square.

EXPLANATION.—

$$\begin{aligned} 3^4 &= 3 \times 3 \times 3 \times 3 \\ &= 3^2 \times 3^2 \\ &= \text{the square of } 3^2 \end{aligned}$$

The sixth power of a number is the square of its cube, or the cube of its square.

EXPLANATION.—

$$\begin{aligned} 5^6 &= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \\ &= 5^2 \times 5^2 \times 5^2 \\ &= \text{the cube of } 5^2 \end{aligned}$$

$$\begin{aligned} \text{Also } 5^6 &= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \\ &= 5^3 \times 5^3 \\ &= \text{the square of } 5^3 \end{aligned}$$

EXAMPLES.—XXII.

1. Square the numbers 15, 35, 55, 115, 135, 295, 1995, 30,005.
2. Square the numbers 325, 525, 1225, 1625, 30,025.
3. Find the fourth power of the numbers 15, 35, 55, 75.

4. Find the cubes of the following quantities correct to four significant figures: 20·283, 34·8721, 102·23, 1225, 116·25.

5. Show that if a quantity be squared three times, we obtain its eighth power.

6. Show that if a quantity is cubed twice, we obtain its ninth power.

36. "Evolution" is the reverse process to involution; that is to say, instead of being given a quantity and being required to find a certain power of it, *we are given the power of the quantity, and are required to find the quantity itself.*

EXAMPLE (1).—*Find the quantity whose square is* 49. *Answer*, 7; because $7 \times 7 = 49$.

EXAMPLE (2).—*Find the quantity whose cube is* 64. *Answer*, 4; because $4 \times 4 \times 4 = 64$.

This process of evolution is more often described as the process of "**extracting a root,**" the term "**root**" being the opposite of the term "power."

Thus the "**square root**" of a number is that number which, when squared, gives the original number.

The "**cube root**" of a number is that number which, when cubed, gives the original number. And so on.

EXAMPLE (1).—*Find the cube root of* 125. *Answer*, 5; because $5 \times 5 \times 5 = 125$.

EXAMPLE (2).—*Find the fifth root of* 243. *Answer*, 3; because $3 \times 3 \times 3 \times 3 \times 3 = 243$.

EXAMPLE (3).—*Find the square root of* ·16. *Answer*, ·4; because $·4 \times ·4 = ·16$.

The symbol $\sqrt{}$ or $\surd$ is used to represent the term *root;* a small figure placed near the top of the root-sign indicates which root is required.

Thus $\sqrt[2]{25}$ means the square root (or second root) of 25; $\sqrt[5]{243}$ means the fifth root of 243. But **if no figure is given to indicate which root is required, it is intended to mean square root.**

Thus the square root of 16 may be written $^2\sqrt{16}$, but is usually written $\sqrt{16}$ or $\surd 16$.

EXAMPLES.—XXIII.

1. Find the square roots of 9, 36, 81, 144, 121, 64, 25, ·09, ·0036, ·0121.

2. Find the cube roots of 27, 64, 216, 512, 1000, 125, 343.

3. Find the cube roots of ·027, ·216, ·001, ·000125, ·343.

4. Find the fourth roots of 81, 625, 256, 16.

5. Give the values of $\sqrt[3]{27}$, $\sqrt{121}$, $\sqrt[5]{32}$, $\sqrt[7]{128}$, $\sqrt[3]{216}$, $\sqrt{·000004}$, $\sqrt{·000049}$, $\sqrt{1·21}$.

37. Extraction of square root.—If a number is too large for us to be able to guess its square root, the following rule must be used. The rule is too complicated to quote in words, and is best learnt by following worked examples.

EXAMPLE (1).—*Find the square root of* 5184.

We first mark off the figures in pairs, *starting from the right.*

There is no square root of 51.

The *nearest number below* 51 which has a square root is 49, whose square root is 7. We place 7 in the root and subtract 49 from 51, leaving 2. We bring down the next pair of figures, giving 284.

```
    51,84(72
    49
14,2) 284
      284
```

We double the figure already in the root (that is 7), giving 14; we place 14 to the left of 284; 14 is called the "*trial divisor.*"

We divide 14 into 28, giving 2; we place the 2 in the root, and also *after* the 14. 142 is called the "*true divisor.*"

We multiply the 142 by the 2 in the root, and subtract the result from the 284, giving the remainder 0.

$$\text{Thus } \sqrt{5184} = 72$$

EXAMPLE (2).—*Find the square root of* 603·6849.

When there is a decimal point, we mark off in pairs *from the decimal point, both to the right and to the left.*

The first step is as before; giving 2 in the root, and a remainder 2; bring down 03.

Doubling the 2 we get trial divisor 4. 4 divided into 20 goes 5; but *if we place* 5 *in the root and after the* 4, *we shall find that we have to subtract* 225 *from* 203; thus 5 is too great, and we try the 4; giving true divisor 44, and remainder 27.

```
      6,03·68,49(24·57
      4
   4,4)203
       176
  48,5) 2768
        2425
 490,7) 34349
        34349
```

In bringing down 68 we *pass the decimal point*, and we then *place the decimal point in the root.*

Doubling the number at present in the root, viz. 24, we obtain the trial divisor 48 (we obviously obtain the same if we bring down the last true divisor, doubling the last figure only).

Dividing 48 into 276 we obtain 5; we place 5 in the root, and also after the 48, giving the true divisor 485; we multiply 485 by 5, and subtract from 2768, and bring down the next pair of figures.

We obtain the next trial divisor either by doubling the number in the root, viz. 245, or by bringing down the last true divisor with its last figure doubled; and continue the process as before.

NOTE.—If the given number is a decimal commencing with cyphers, place one cypher after the decimal point in the root for each pair of cyphers after the decimal point in the given number.

EXAMPLE (3).—*Extract the square root of* ·0001564311 *correct to six significant figures.*

```
        ·00,01,56,43,11(·01250724 . . .
             1
        2,2) 56
             44
       24,5)1243
            1225
    250,0,7)  181100
              175049
     25014,2)  605100
               500284
    250144,4)10481600
```

In this example the following points are important :—

(*a*) The third trial divisor is 250 ; dividing 250 into 181 we obtain 0, *which is placed both in the root and after the trial divisor ;* we bring down 00, giving 181100, and our trial divisor is now 2500, which, when divided into 18110 gives 7 ; etc.

(*b*) If the process does not work out exactly when the last figures in the top line are brought down, it will *never work out exactly*, however far we carry it ; *nor* will the decimals obtained in the root *ever recur.*

(*c*) It was necessary to obtain the seventh significant figure in the root in order to see whether it would be more correct to reckon the sixth significant figure as 2 or 3. Since the seventh figure is 4, we quote the answer as ·0125072.

Roots which do *not work out exactly* are termed "**surds.**"

Thus $\sqrt{3}$ is a surd ; if we extract the square root of 3·0000 . . . correct to six significant figures we shall obtain 1·73205 ; squaring this we obtain 2·9999972025, which is, of course, for all practical purposes, equal to 3.

EXAMPLES.—XXIV.

Extract the square roots of—

1. 15625. **2.** 82369. **3.** 14976900. **4.** 1361610000.
5. ·194481. **6.** 351·225081. **7.** ·0000038025. **8.** ·000000164025.
9. 1834·22 correct to 5 significant figures.
10. ·00192 correct to 4 significant figures.

38. EXPLANATION.—(The student who is unacquainted with Algebra must postpone this explanation till after he has read Chapter VIII.)

Consider Example (2), § 37, which for convenience of reference is reprinted on this page.

We guess 2 as the first figure in the root ; this 2 represents 20, and in taking that as the first figure we are assuming that the *actual square root lies between* 20 *and* 30, which is obvious, since the quantity given *lies between* 20^2 *and* 30^2, *i.e.* 400 *and* 900.

```
        6,03·68,49(24·57
        4
  4,4)203
       176
  48,5) 2768
        2425
  490,7) 34349
         34349
```

The first step is equivalent to subtracting the square of 20, viz. 400, from 603·6849, leaving the remainder 203·6849.

In the next step we obtain 4 as the second figure in the root (by a process which we will justify later). We now wish to obtain the remainder when the square of 24 has been subtracted from 603·6849 ; but we have already subtracted the square of 20.

Let a represent 20, and b represent 4 ; then by algebra $(a + b)^2 = a^2 + 2ab + b^2 = a^2 + (2a + b)b$. Now, as we have already subtracted the quantity represented by a^2, *it remains to subtract the quantity represented by* $(2a + b) \times b$. But the true divisor (viz. 44), from the method of its formation, is equivalent to $2a + b$, and b represents 4 ; thus *the true divisor multiplied by* 4 *is the extra quantity which must be subtracted from the previous remainder* 203·6849, giving us the new remainder 27·6849.

Similarly, in the third step, we have now subtracted the square of 24, and we require the correct remainder after subtracting the square of 24·5. Let a represent 24, and b represent ·5 ; then, as before, we must subtract $(2a + b)b$ from the last remainder to obtain the required remainder ; but

$2a + b$ represents 48·5, and b represents ·5; hence we subtract the product of these two numbers from the last remainder 27·6849 to obtain the new remainder 3·4349; and so on.

It remains to justify the process by which we obtain the successive figures in the root.

Let us again consider the second step in the process. We have a trial divisor 4 (really 40), and a remainder 203·6849, and the figure required in the units' place must obviously be as large as possible consistently with the requirement that the product of this figure and the true divisor is not greater than 203. Remembering that the true divisor is something greater than 40, the best method to find this figure is obviously to divide the 40 into the 203; the number so obtained *may be too large* (as is actually the case in this step), *but it cannot be too small.*

39. Contracted method for Square Root.—To extract a square root correct to a given number of significant figures.

RULE.—(a) **More than half the required number of significant figures must be extracted by the ordinary process**; for instance, if five significant figures are required three must be extracted by the ordinary process; if six are required, four must be extracted by the ordinary process.

(b) **Bring down one figure instead of two; form the next trial divisor; and proceed by the method of contracted division.**

EXAMPLE.—*Find the square root of* 10 *correct to seven significant figures.*

We extract the square root to four significant figures by the ordinary method.

Then bring down 0, instead of 00, and form the next trial divisor 6324; and proceed by contracted division.

The work may be still further contracted by doing multiplications and subtractions in one step, as in § 31.

As the eighth significant figure is 6, we should quote the result as 3·162278.

```
          10 00(3·1622776
           9
   6,1)   100
           61
  62,6)  3900
         3756
 632,2) 14400
        12644
6,3,2,4) 17560
         12648
          4912
          4427
           485
           442
            43
            38
             5
```

EXPLANATION.—Let a represent 3·162, and let x represent the remainder of the root; then, as we have shown before, 6·324 is represented by $2a$, and the remainder at this stage, viz. ·001756, is represented by $2ax + x^2$. It follows that the result of dividing ·001756 by 6·324 corresponds to $(2ax + x^2) \div 2a$, which is equal to $x + \frac{x^2}{2a}$. Now, x *is less than* ·001, since the first figure of x is in the fourth decimal place; therefore x^2 *is less than* $(·001)^2$, i.e. *than* ·000001. Also a is greater than 1; *therefore* $\frac{x^2}{2a}$ *is less than* $\frac{·000001}{2}$, or ·0000005. Thus $x + \frac{x^2}{2a}$ is the same as x as far as the first six decimal places are concerned. That is to say, the first seven significant figures in the result are correct, the eighth is doubtful.

NOTE.—The general rule which applies to this process is that the number of significant figures which are given correctly by the contracted division is one less than the number which have been extracted by the ordinary square root process.

EXAMPLES.—XXV.

Evaluate correctly to seven significant figures—

1. $\sqrt{2}$. **2.** $\sqrt{3}$. **3.** $\sqrt{5}$. **4.** $\sqrt{6}$. **5.** $\sqrt{7}$. **6.** $\sqrt{11}$.
7. $\sqrt{3 \cdot 14159}$. **8.** $\sqrt{1414 \cdot 2135}$. **9.** $\sqrt{\frac{246}{169}}$. **10.** $\sqrt{\frac{100000}{17}}$.

40. We have already shown that if we square a quantity twice we obtain the fourth power; and that if we cube it and square the result we obtain the sixth power. (See § 35.)

It follows that we obtain the **fourth root** of a quantity by taking the **square root of its square root**; and the **sixth root** of a quantity by taking the **cube root of its square root.**

Similarly, the **square root of the fourth root will give the eighth root.**

EXAMPLE.—*Find* $\sqrt[6]{15625}$.

By the method of § 37 we find the square root of 15625 to be 125.
The cube root of 125 is obviously **5.**

EXAMPLES.—XXVI.

Evaluate—

1. $\sqrt[4]{390625}$. **2.** $\sqrt[4]{83521}$. **3.** $\sqrt[6]{531441}$.

Evaluate correct to four significant figures—

4. $\sqrt[4]{27}$. **5.** $\sqrt[4]{38}$. **6.** $\sqrt[8]{10}$. **7.** $\sqrt[8]{50}$.

CHAPTER V.

ON UNITS AND MEASUREMENT.

41. THE process of measurement consists essentially of *comparing the quantity to be measured with some standard quantity of the same kind.*

This standard quantity is called the **unit**, and the ratio which the quantity to be measured bears to the standard quantity is called its **measure**; thus in the quantity $3\frac{1}{4}$ yards, the **unit** is the **yard**, and the **measure** is $3\frac{1}{4}$.

42. We shall assume that the student understands the arithmetical processes of reduction; but for the sake of convenience of reference we will give the more important tables of weights and measures.

Table of Units of Length.

12 inches	= 1 foot
3 feet	= 1 yard
$5\frac{1}{2}$ yards	= 1 pole
40 poles	= 1 furlong
8 furlongs	= 1 mile
3 miles	= 1 league

Remember that—

1 mile	= 1760 yards
,,	= 63,360 inches

Table of Units of Area.

144 square inches	= 1 square foot
9 square feet	= 1 square yard
$30\frac{1}{4}$ square yards	= 1 square pole
40 square poles	= 1 rood
4 roods	= 1 acre
640 acres	= 1 square mile

Remember that—

1 acre	= 4840 square yards

Surveyors measure land by means of a chain of length 22 yards (the length of a cricket pitch), which is divided into 100 links of equal length. Remember that—

10 chains	= 1 furlong
10 square chains	= 1 acre
1 nautical mile	= 2029 yards
1 nautical mile	= 1000 fathoms

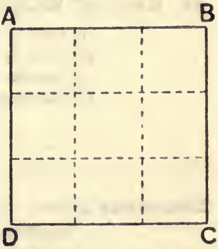

FIG. 1.

The connection between the corresponding portions of the tables in long and square measure is obvious from the accompanying diagram (Fig. 1).

If **ABCD** represents a **square yard**, that is, a square each of whose sides is a yard, and if the dotted lines divide each side into three feet, as represented; then *each of the small squares represents a square foot*, being a foot long and a foot broad. But we have *three rows* of small squares, and *each row contains three squares.* Therefore the total number of squares is nine ; that is, 1 sq. yard is equal to 9 sq. feet.

To show that 1 sq. pole = $30\frac{1}{4}$ sq. yards ; we know that in long measure 1 pole = $5\frac{1}{2}$ yards ; *i.e.* 1 pole = 11 half-yards. Hence, by a similar reasoning to the preceding—

1 *square pole* = 121 "*square half-yards,*"

where a "square half-yard" means a square each of whose sides is half a yard long. But since—

1 yard	= 2 half-yards
∴ 1 *square yard*	= 4 *square half-yards*
∴ 121 square half-yards	= $\frac{121}{4}$ square yards
	= $30\frac{1}{4}$ square yards

43. Tables of Units of Volume, or "Capacity."

1728 cubic inches = 1 cubic foot	4 gills = 1 pint
27 cubic feet = 1 cubic yard	2 pints = 1 quart
	4 quarts = 1 gallon

1 gallon = 277·274 cubic inches

The connection between the tables in long and cubic measure is explained by the accompanying diagram (Fig. 2).

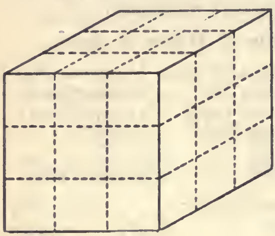

FIG. 2.

If the figure represents a **cubic yard** (that is, a block whose length, breadth, and height are each 1 yard), and if this be divided up into cubic feet as represented in the diagram ; then there are obviously *nine small cubes* in the top layer, and *three such layers* make up the complete cube. Therefore 1 cub. yard contains 27 cub. feet.

Note that each face of the cube is a square yard, and is divided into 9 sq. feet.

In general, we measure the volumes of solids by the first table, and of liquids by the second ; but the rule is by no means absolute.

44. Units of Weight.—"Avoirdupois" Table.

16 ounces (oz.)	= 1 pound (lb.)
28 pounds	= 1 quarter (qr.)
4 quarters	= 1 hundredweight (cwt.)
20 hundredweights	= 1 ton

7000 grains	= 1 lb.
14 lbs.	= 1 stone
2240 lbs.	= 1 ton
112 lbs.	= 1 cwt.

Remember also—

1 gallon of water weighs 10 lbs
1 cubic foot of water weighs 997·1 ozs.

In ordinary cases, we take 1000 ozs. as sufficiently correct for the weight of 1 cub. foot of water.

EXAMPLE (1).—*Reduce* 2 *cwt.* 1 *qr.* 12 *lbs.* 5 *ozs. to the decimal of a ton.*

16)5 . . . . . . ozs.
28)12·3125 . . . lbs.
4)1·4397 . . . qrs.
20)2·3599 . . . cwts.
·1179 . . . ton.

EXPLANATION.—We reduce the 5 ozs. to the decimal of 1 lb. by dividing by 16; this gives ·3125 lb. Thus we have altogether 12·3125 lbs.

We reduce the 12·3125 lbs. to the decimal of 1 qr. by dividing by 28; this gives ·4397 qr. Thus we have altogether 1·4397 qrs. And so on; working to any required degree of accuracy.

EXAMPLE (2).—*Reduce* 8 *poles* 4 *yards* 2 *feet to the decimal of* 1 *furlong.*

3)2 . . . . . . ft.
4·66666 . . . yards
2
11)9·33333 . . . half-yards
40)8·84848 . . . poles
·22121 . . . furlong

EXPLANATION.—We reduce yards to half-yards by multiplying by 2; we reduce half-yards to poles by dividing by 11.

EXAMPLE.—*Reduce* ·013876 *of a mile to furlongs, poles, etc.*

·013876 mile
8
·111008 furlong
40
4·440320 poles
11
2)4·84352 half-yards
2·42176 yards
3
1·26528 feet
12
3·18336 inches

Result, 4 poles 2 yards 1 foot 3·18 inches.

EXPLANATION.—To reduce the miles to furlongs, multiply by 8; to reduce the furlongs to poles, multiply by 40; the 4 *poles are reduced no farther;* the ·44032 of a pole is reduced to yards by multiplying by $\frac{11}{2}$; the 2 yards are reduced no farther; the ·42176 of a yard is reduced to feet; and so on.

EXAMPLES.—XXVII.

(Work Questions 1–5 correct to five significant figures.)

1. Reduce 4 furlongs 17 poles 2 yards to the decimal of a mile.
2. Reduce 5 poles 1 yard 7 inches to the decimal of a mile.
3. Reduce 29 yards 11 inches to the decimal of a furlong.
4. Reduce 3 cwt. 10 lbs. 8 ozs. to the decimal of a ton.
5. Reduce 25 sq. poles 4 sq. yards to the decimal of an acre.
6. Reduce ·2457 mile to furlongs, etc.
7. Reduce ·3817 acre to roods, etc.
8. Reduce ·2379 ton to cwts., etc.
9. Reduce ·2954 sq. yard to square feet and square inches.
10. Reduce ·8729 cub. yard to cubic feet and cubic inches.

45. The Metric System.—This is undoubtedly the best system of weights and measures. The principle of the system is that the ratios of the various units are *all powers of* 10.

The student must learn by heart the following list of prefixes :—

milli-	= 1-thousandth
centi-	= 1-hundredth
* **deci-**	= 1-tenth
† **deca-**	= ten
hecto-	= a hundred
kilo-	= a thousand
myria-	= ten thousand

The use of these prefixes is as follows :—

The unit of length is a **metre.**

[1 metre = 39·3708 inches.]

Then a **deci**-metre $= \frac{1}{10}$ metre
a **hecto**-metre = 100 metres, etc. . . .

The unit of weight is a **gramme.**

[1 gramme = ·002204 lb.]

Then a **kilo**-gramme = 1000 grammes
a **milli**-gramme $= \frac{1}{1000}$ gramme

The unit of volume or "capacity" is a **litre.**

[1 litre = 1·765 pints.]

Then a **deca**-litre = 10 litres, etc.

The unit of area in land-surveying is the **are.**

[1 are = 100 square metres = ·02471 acre.]

Then a **hect**-are = 100 ares, etc.

For the measurement of small areas, we use **square centimetres, square metres,** etc.

The student must remember that **1 square decametre** is *not* equal to 10 square metres. Just as 1 sq. foot = 144 sq. inches (*not* 12 sq. inches), so

1 *square decametre* = 100 *square metres*
1 *square decimetre* $= \frac{1}{100}$ *square metre;* etc.

* Soft *c*. † Hard *c*, sometimes spelt *deka-*.

For measuring timber, etc., the unit of volume is the **stère.**

[1 stère = 1 cubic metre = 35·32 cubic feet.]

For ordinary purposes we use the cubic centimetre, cubic metre, etc.; and for the volumes of liquids or gases we use the litre.

Note again that—

1 *cubic decimetre* = $\frac{1}{1000}$ *cubic metre;* etc.

The relations between these different systems of units are very important, and are as follows :—

1 gramme = weight of 1 cubic centimetre of water
1 litre = 1 cubic decimetre = 1000 cubic centimetres

The following **abbreviations** are in general use :—

mm. for millimetre
cm. „ centimetre
kilom. „ kilometre
c.c. „ cubic centimetre
gm. „ gramme
kilog. „ kilogramme

EXAMPLE (1).—*Reduce* 32578·26 *decimetres to kilometres.*

1 kilom. = 1000 metres
and 1 metre = 10 decimetres
∴ 1 kilom. = 10,000 decimetres

Therefore to reduce decimetres to kilometres we must divide by 10,000; *i.e.* we move the decimal point 4 places to the left;

thus we obtain 3·257826 kiloms.

EXAMPLE (2).—*Find the weight of* 15 *kilolitres of water.*

15 kilolitres = 15,000 litres
But 1 litre of water weighs 1 kilog.
[For 1 litre = 1000 c.c., and 1 c.c. weighs 1 gm.]
∴ required weight = 15,000 kilog.

EXAMPLES.—XXVIII.

1. Reduce ·00358 kilometre to millimetres.
2. Reduce 25·72 centilitres to decalitres.
3. Reduce 00549 decigramme to hectogrammes.
4. Reduce 528·372 kilogrammes to decagrammes.

Work the following examples correct to four significant figures:—

5. Reduce 1000 kilog. to the decimal of a ton.
6. Reduce 1 kilom. to the decimal of a mile.
7. Find the weight of 15·32 cub. metres of water in tons.
8. Find the weight of 150 c.c. of alcohol in ounces, if a litre of alcohol weighs 793 gms.
9. Find the volume of 15 ozs. of water in litres.
10. Find the ratio of 2·57 pints to 3258 c.c.

CHAPTER VI.

ON THE MEANING OF ALGEBRAIC SYMBOLS.

46. In Algebra we represent quantities by letters. The advantage of this method is perhaps, at first, hard to understand; but the following may give some slight idea of its value.

If we wish to calculate the length of paper required to paper the walls of a room, we must use the following arithmetical rule :—

Add twice the length of the room to twice its breadth; multiply the result by the height of the room; divide this result by the width of the paper.

But if we represent the length, breadth, and height of the room by the letters l, b, and h respectively, and the width of the paper by w,—we can express this rule by means of the following algebraic "formula" :—

$$(2l + 2b) \times h \div w$$

In this formula, "$2l + 2b$" denotes that twice the length is to be added to twice the breadth; "$\times h$" denotes that the result is to be multiplied by the height (the brackets denote that it is the whole result $2l + 2b$ which is to be multiplied by h, and not $2b$ only); "$\div w$" denotes that this result is to be divided by the width.

The student will find by experience that most rules in scientific or practical calculations are far more easily remembered in the concise form of an algebraical "formula."

A formula may be defined as an arithmetical rule expressed by algebraical symbols.

47. The process of multiplication is so frequent in Algebra that the **ordinary sign of multiplication, viz. ×, is usually omitted between two algebraical quantities that are to be multiplied together**; it is always to be understood that if two or more quantities are placed side by side, without any sign between them, they are to be multiplied.

Thus ab means a multiplied by b.
$5x$ means 5 multiplied by x.
$26abc$ means $26 \times a \times b \times c$.

Note that 26 means 2 tens and 6 units; but that ab has quite a distinct meaning, viz. $a \times b$.

Sometimes a full stop is placed between quantities which are to be multiplied; thus $6 . a . b . c$ means $6 \times a \times b \times c$; but this is more often used for numbers only; thus $1 . 2 . 3 . 4$ means $1 \times 2 \times 3 \times 4$, *i.e.* 24.

A quantity represented by a group of algebraic symbols is called an **algebraical "expression."**

EXAMPLE (1).—*If a represents* 4, *b represents* 5, *and c represents* 7, *find the value of the expressions* $3ab$, $6bc$, $27abc$.

$$3ab = 3 \times 4 \times 5 = 60$$
$$6bc = 6 \times 5 \times 7 = 210$$
$$27abc = 27 \times 4 \times 5 \times 7 = 3780$$

When we wish to represent division, we place the dividend above the divisor, with a line between them; thus $\frac{a}{b}$ means $a \div b$; just as in arithmetic $\frac{5}{7}$ may be interpreted as $5 \div 7$.

Sometimes, for the sake of convenience in printing, we write the two quantities on the same level (the dividend first) with a slanting line, called the "**solidus,**" between them; thus $a/7$ means $a \div 7$, $5b/cd$ means $\frac{5 \times b}{c \times d}$.

EXAMPLE (2).—*If* $a = 8$, $b = 4$, $c = 3$, *find the value of the expressions* $\frac{ac}{b}$, $\frac{3a}{bc}$, $\frac{14bc}{a}$, $3ab/4c$.

$$\frac{ac}{b} = \frac{8 \times 3}{4} = 6$$
$$\frac{3a}{bc} = \frac{3 \times 8}{4 \times 3} = 2$$
$$\frac{14bc}{a} = \frac{14 \times 4 \times 3}{8} = 21$$
$$3ab/4c = \frac{3 \times 8 \times 4}{4 \times 3} = 8$$

Note that if we multiply any quantity by 0, we obtain 0 as the product; thus $4 \times 5 \times 0 \times 3 = 0$.

EXAMPLES.—XXIX.

If $a = 3$, $b = 5$, $c = 2$, evaluate—

1. $4ab$. **2.** $3ac$. **3.** $12abc$. **4.** $25bc$. **5.** $\frac{3b}{a}$.

6. $\frac{5c}{ba}$. **7.** $8ab/5c$. **8.** $10ac/3b$. **9.** $14ac/21b$.

If $p = 5$, $q = 4$, $r = 0$, $s = 10$, evaluate—

10. $3pq$. **11.** $10qrs$. **12.** $8pq/s$. **13.** $26rs/9pq$.

14. $\frac{12ps}{25q}$. **15.** $\frac{25q}{ps}$. **16.** $\frac{7prs}{8q}$.

48. Powers have the same meaning in Algebra as in Arithmetic (see § 33).

Thus a^3 denotes $a \times a \times a$

bc^2 denotes $b \times c^2$, *i.e.* $b \times c \times c$

(Note that the sign of the square refers to the c only.)

$5b^3c^2a$ denotes $5 \times b \times b \times b \times c \times c \times a$

EXAMPLES.—If $a = 5$, $b = 3$, $c = 1$, $d = 0$, evaluate $3a^2b$, $5abc^3$, $2a^4b^2c^2$, $6a^2bd^3$, $108a^3c/25b^2$.

$$3a^2b = 3 \times a \times a \times b = 3 \times 5 \times 5 \times 3 = 225$$
$$5abc^3 = 5 \times a \times b \times c \times c \times c = 5 \times 5 \times 3 \times 1 \times 1 \times 1 = 75$$
$$2a^4b^2c^2 = 2 \times a \times a \times a \times a \times b \times b \times c \times c = 2 \times 5 \times 5 \times 5 \times 5 \times 3 \times 3 \times 1 \times 1 = 11{,}250$$
$$6a^2bd^3 = 6 \times a \times a \times b \times d \times d \times d = 6 \times 5 \times 5 \times 3 \times 0 \times 0 \times 0 = 0$$
$$108a^3c/24b^2 = \frac{108 \times a \times a \times a \times c}{25 \times b \times b} = \frac{108 \times 5 \times 5 \times 5 \times 1}{25 \times 3 \times 3} = 60$$

EXAMPLES.—XXX.

If $a = 6$, $b = 9$, $c = 4$, $d = 1$, $e = 0$, evaluate—

1. $30a^2d$. **2.** $5ab^2cd^4$. **3.** a^3bc^2. **4.** b^3cd^5. **5.** abc^5e^2.
6. $20a^2bc^4$. **7.** a^2b^2/cd^2. **8.** $3a^2b/4c$. **9.** $10ab^5e/c^2d$.
10. $120a^4/b^2c^2$. **11.** abe/c^4d. **12.** $3ac/bd^4$.

49. The letters need not necessarily represent integers; they may represent vulgar or decimal fractions or surds.

EXAMPLE (1).—*Evaluate* $\frac{3ab^2}{c^3}$, *where* $a = 4$, $b = \frac{3}{5}$, $c = \frac{3}{4}$.

$$\frac{3ab^2}{c^3} = \frac{3 \times 4 \times \frac{3}{5} \times \frac{3}{5}}{\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}} = \frac{3 \times 4 \times 3 \times 3 \times 4 \times 4 \times 4}{3 \times 3 \times 3 \times 5 \times 5}$$
$$= \frac{256}{25} = 10{\cdot}24$$

EXAMPLE (2).—*Evaluate* $3\sqrt{a}/b^2$, *correct to four significant figures; where* $a = 5{\cdot}2$, $b = 2{\cdot}36$.

$$3\sqrt{a}/b^2 = \frac{3 \times \sqrt{5{\cdot}2}}{2{\cdot}36 \times 2{\cdot}36}$$

By the method of § 39, we find that $\sqrt{5{\cdot}2} = 2{\cdot}28035 \ldots$;

$$\therefore\ 3 \times \sqrt{5{\cdot}2} = 6{\cdot}84105 \ldots$$

Also $2{\cdot}36 \times 2{\cdot}36 = 5{\cdot}5696$.

And by the method of § 31—

$$6{\cdot}84105 \div 5{\cdot}5696 = 1{\cdot}228$$

EXAMPLES.—XXXI.

Evaluate correct to four significant figures—

1. $\frac{4}{3}ab^3$, where $a = 3{\cdot}1416$, and $b = 2{\cdot}25$.
2. $\frac{1}{3}abc^2$, where $a = 3{\cdot}1416$, $b = 2{\cdot}38$, $c = 1{\cdot}15$.
3. $p^2q/3xy^3$, where $p = \frac{2}{5}$, $q = \frac{1}{7}$, $x = 11$, $y = 2{\cdot}4$.
4. $3lm/k^3h^2$, where $l = \sqrt{3}$, $m = \sqrt{5}$, $k = 3$, $h = 1{\cdot}5$.
5. $\sqrt{2ab^2c^3d}$, where $a = 1{\cdot}2$, $b = 3{\cdot}4$, $c = 5{\cdot}6$, $d = 7{\cdot}8$. (Evaluate $2ab^2c^3d$, and find the square root of the result.)

50. The sign + (called "**plus**") denotes **addition**; the sign − (called "**minus**") denotes that the quantity **after** it is to be **subtracted** from the quantity **before** it. The sign ∼ denotes that we are to find **the difference between** the two quantities which it connects. Thus—

$$5 + 7 = 12$$
$$7 - 5 = 2$$

$$25 \sim 30 = \text{the difference between } 25 \text{ and } 30 = 5$$

Again, the expression $7 - 5 + 8 - 6 + 2 + 1$ denotes that we are to subtract 5 from 7; add 8 to the result; subtract 6 from the result; add 2 to the result; add 1 to the result. This gives ultimately 7.

It is, however, more usual to arrive at the result by adding 7, 8, 2, and 1; adding 5 and 6; and subtracting the second result, viz. 11, from the first, viz. 18.

The order in which the additions and subtractions are performed does not matter, provided that we *subtract all the quantities which are preceded by a minus sign.*

51. The Commutative Law.—The law that a **series of operations may be performed in any order,** without altering the final result, is called the **Commutative Law.**

We have just seen that the **Commutative Law applies to a series of additions and subtractions**; it is known also to apply to a **series of multiplications and divisions.** Thus—

$$28 \div 7 \times 3 = 4 \times 3 = 12$$
$$28 \times 3 \div 7 = 84 \div 7 = 12$$

But the Commutative Law **does not apply when additions or subtractions (or both) are combined with multiplications or divisions (or both)**; *e.g.* in $7 \times 4 + 3$; if we first multiply 7 by 4, and then add 3, we obtain 31; if we first add 4 and 3, and then multiply by 7, we obtain 49; if we first add 7 and 3, and then multiply by 4, we obtain 40.

Seeing that the result, in a case of this kind, depends upon the order in which the operations are performed, we must make some definite rule as to the order. We agree, then, to work by the following

RULE.—**Perform first the multiplications and divisions; and afterwards the additions and subtractions.**

Thus
$$\begin{aligned} & 7 \times 4 + 16 \div 2 \times 3 - 6 \times 9 \div 18 + 2 \\ = {} & 28 + 8 \times 3 - 54 \div 18 + 2 \\ = {} & 28 + 24 - 3 + 2 \\ = {} & 51 \end{aligned}$$

Again, in $\frac{4}{3} \times \frac{4}{5} - \frac{1}{8} \div \frac{2}{7} \times \frac{4}{3} + \frac{2}{5}$

$$\tfrac{4}{3} \times \tfrac{4}{5} = \tfrac{16}{15}; \quad \tfrac{1}{8} \div \tfrac{2}{7} \times \tfrac{4}{3} = \tfrac{7}{12}$$
$$\text{thus we have } \tfrac{16}{15} - \tfrac{7}{12} + \tfrac{2}{5} = \tfrac{53}{60}$$

The quantities which are separated from one another by the + and − signs, are called the "**terms**" of the expression. Thus a "**term**" is either a single quantity which is to be added or subtracted, or a group of two or more quantities connected by × or ÷ signs, the result of which is to be added or subtracted. In the last expression, the separate "terms" are $\frac{4}{3} \times \frac{4}{5}$; $\frac{1}{8} \div \frac{2}{7} \times \frac{4}{3}$; $\frac{2}{5}$.

EXAMPLE (1).—*Evaluate* $2ab + 3ac - 4bc$; *where* $a = 3$, $b = 4$, $c = 1$.

$$\begin{aligned} 2ab + 3ac - 4bc &= 2 \times 3 \times 4 + 3 \times 3 \times 1 - 4 \times 4 \times 1 \\ &= 24 + 9 - 16 \\ &= 17 \end{aligned}$$

EXAMPLE (2).—*Evaluate* $\cdot04ab + \cdot27bc - \cdot15ac$; *where* $a = 3$, $b = 4$, $c = \cdot28$.

$$\begin{aligned} \cdot04ab + \cdot27bc - \cdot15ac &= \cdot04 \times 3 \times 4 + \cdot27 \times 4 \times \cdot28 - \cdot15 \times 3 \times \cdot28 \\ &= \cdot48 + \cdot3024 - \cdot126 \\ &= \cdot6564 \end{aligned}$$

EXAMPLE (3).—$\frac{3}{4}ab \div c - 20a^2c \div b + \frac{a^3bc}{10}$; *where* $a = 5$, $b = \cdot04$, $c = \cdot003$

$$\begin{aligned} &\tfrac{3}{4}ab \div c - 20a^2c \div b + \frac{a^3bc}{10} \\ &= \tfrac{3}{4} \times 5 \times \cdot04 \div \cdot003 - 20 \times 5 \times 5 \times \cdot003 \div \cdot04 + 5 \times 5 \times 5 \times \cdot04 \times \cdot003 \div 10 \\ &= 50 - 37\cdot5 + \cdot0015 \\ &= 12\cdot5015 \end{aligned}$$

EXAMPLES.—XXXII.

If $a = 5$, $b = 6$, $c = 3$, $d = 0$, evaluate—

1. $2ab + 3cd + 6ac + 3bc$. **2.** $3a^2 + 2bc - 20cd$.

3. $4a^2b - 2bc + 3ac^2 - 2abc$.

4. $4a^2 \times b \div c - 15c + ad \div c^2 - 25b^2 \div ac$.

5. $\cdot03a^2b^2 + 6a \times b \div c - \frac{a^2b^2}{5c^2} - a^3cd \div b^2$.

If $a = \cdot4$, $b = \cdot3$, $c = \cdot02$, $d = \cdot0357$, evaluate correct to four significant figures—

6. $3a^3b + 2a^2bc - 4c^2d$. **7.** $3a \times b^2c + 2a^2b \div c - 3cd$.

8. $3\sqrt{\frac{a}{10}} \div bc - a^2 \times b \div c + \sqrt{a^2bc}$.

9. $\cdot0346a + \cdot2587bc - \cdot28b^2c + c \div d$.

10. $\frac{a^2 - b^2}{2c - d^2}$. (First evaluate the numerator; then the denominator; then divide.)

11. $\frac{a + b}{c + 2d} - \frac{a^2 + 2b}{c} + \frac{a^3}{cd}$. **12.** $ab^2 + \frac{ab}{4cd} + \sqrt{5c}$.

52. On the use of Brackets.—If any portion of an expression is enclosed in **brackets**, this is intended to indicate that this part is to be **evaluated separately**, and the **result** combined with the rest of the expression in whatever manner may be indicated.

The use of brackets is particularly important, and the following examples should be very carefully studied :—

EXAMPLE (1).—$(7 + 5) \div 3$.

(i.) Evaluate the bracket * : $7 + 5 = 12$.

(ii.) $12 \div 3 = 4$.

Note the difference between this, and the same expression without the brackets—

$$7 + 5 \div 3 = 7 + \tfrac{5}{3} = 8\tfrac{2}{3}. \quad \text{(See § 51.)}$$

EXAMPLE (2).—$(8 + 3 \times 4) \div (3 + 16 \div 8)$.

(i.) Evaluate the first bracket : $8 + 3 \times 4 = 8 + 12 = 20$.

(ii.) Evaluate the second bracket : $3 + 16 \div 8 = 3 + 2 = 5$.

(iii.) $20 \div 5 = 4$

If the brackets were omitted we should have—

$$8 + 3 \times 4 \div 3 + 16 \div 8 = 8 + 4 + 2 = 14$$

EXAMPLE (3).—*Evaluate* $12a(c + d) - (20a - 5b)$; *where* $a = \cdot 4$, $b = \cdot 5$, $c = \cdot 6$, $d = \cdot 89$.

Note that $12a(c + d)$ means $12 \times a \times (c + d)$; cf. § 47.

(i.) Evaluate $c + d$: $c + d = \cdot 6 + \cdot 89 = 1\cdot 49$.

(ii.) Evaluate $20a - 5b$: $20a - 5b = 20 \times \cdot 4 - 5 \times \cdot 5 = 8 - 2\cdot 5 = 5\cdot 5$.

(iii.) Thus $12a(c + d) - (20a - 5b) = 12 \times \cdot 4 \times 1\cdot 49 - 5\cdot 5 = 7\cdot 152 - 5\cdot 5 = 1\cdot 652$.

EXAMPLE (4).—*Evaluate* $\left(\frac{3a + b}{c} - \frac{a}{b + 3c}\right)\left(\frac{2a}{d} - \frac{b}{c}\right)$; *where the letters have the same values as in Ex.* (3).

Since there is no sign between the brackets, they must be multiplied together (§ 47).

(i) $\frac{3a + b}{c} - \frac{a}{b + 3c} = \frac{1\cdot 2 + \cdot 5}{\cdot 6} - \frac{\cdot 4}{\cdot 5 + 1\cdot 8} = \frac{1\cdot 7}{\cdot 6} - \frac{\cdot 4}{2\cdot 3} = \frac{17}{6} - \frac{4}{23} = \frac{367}{138}$

(ii.) $\frac{2a}{d} - \frac{b}{c} = \frac{\cdot 8}{\cdot 89} - \frac{\cdot 5}{\cdot 6} = \frac{80}{89} - \frac{5}{6} = \frac{35}{534}$

(iii.) $\frac{367}{138} \times \frac{35}{534} = \frac{12845}{73692}$

EXAMPLE (5).—*Evaluate* $(a + b)^2 - (a^2 + b^2)$; *when* $a = 2\cdot 3$, $b = 2\cdot 7$.

(i.) $a + b = 2\cdot 3 + 2\cdot 7 = 5$

(ii.) $\therefore (a + b)^2 = 5^2 = 25$

(iii.) $a^2 + b^2 = 2\cdot 3 \times 2\cdot 3 + 2\cdot 7 \times 2\cdot 7$
$= 5\cdot 29 + 7\cdot 29 = 12\cdot 58$

(iv.) $25 - 12\cdot 58 = 12\cdot 42$

EXAMPLES.—XXXIII.

If $a = 3$, $b = 4$, $c = \frac{1}{2}$, $d = 0$, $e = 2$, evaluate—

1. $a(b + c) - a(b - c)$.

2. $(a + b)(c + d) - (a - b)(c - d)$.

3. $\left(ab - \frac{a}{b}\right) - \left(cd - \frac{d}{c}\right)$.

4. $3\,a^3(b^2 + c) - 2\,c^3(d^2 + e)$.

5. $a^2(b^2 + c^2) - 5c/(d + e)$.

6. $2b\left(\frac{a}{b} + \frac{c}{e} - \cdot 3\right) - 3c\left(\frac{b}{4c} + \frac{d}{e} - \frac{5}{a}\right)$.

---

* The *quantity* enclosed within a pair of brackets is often referred to as a "bracket."

If $a = 1{\cdot}23$, $b = {\cdot}56$, $c = {\cdot}04$, $d = {\cdot}3$, evaluate—

7. $3a(b + c) - b(c + d)$.

8. $4a(b^2 + c) - 2b(5c - 2d^2)$, correct to five significant figures.

9. $2a(b - c)(a^2 + d^2)$, correct to four significant figures.

10. $3\left(\frac{a}{b} + \frac{c}{d}\right) - a \div (b - c)$, correct to four significant figures.

11. $\frac{abc}{5d} + \left(\frac{a}{b} + \frac{c}{d}\right) - \frac{4 \times d}{a + 2b}$.

If $a = {\cdot}2345$, $b = {\cdot}1872$, $c = 2{\cdot}248$, $d = 3{\cdot}347$, evaluate, correct to four significant figures—

12. $3(ab - {\cdot}011) + cd - (d - c)/5a$.

13. $2\left(\frac{a}{b} - \frac{c}{d}\right) + \left(\frac{4b}{a} - \frac{d}{c}\right)$.

14. $2(a^2 - b^2)(c^2 - d)$.

15. $(2\sqrt{a} - \sqrt{b})(\sqrt{c} + \sqrt{d})$.

If $a = 3{\cdot}2$, $b = 2{\cdot}8$, evaluate—

16. $(a + b)^2 - (a^2 + b^2)$.

17. $(a + b)^3 - (a^3 + b^3)$.

18. $(a + b)^4 - (a^4 + b^4)$.

19. $2(a + 2b)^2 - (2a - b)^2$.

20. $3(3a + 2b)^2 - 2(3a - 2b)^2$.

53. Brackets within Brackets.—Sometimes a quantity, some part of which is already within brackets, is placed within another pair of brackets. In this case it is usual to use a different kind of bracket for the outer pair.

In evaluating such quantities we must *find the value of the inner bracket or brackets first;* then, performing on these results the operations indicated, we can *find the value of the outer bracket;* and so on.

EXAMPLE (1).—*Evaluate* $2 + 3\{15 - (4 + 8)\}$.

(i.) Evaluate the bracket $(4 + 8)$: result 12.

(ii.) Evaluate the bracket $\{15 - 12\}$: result 3.

(iii.) $2 + 3 \times 3 = 2 + 9 = 11$.

EXAMPLE (2).—*Evaluate* $(2a + b)\{3c - (2d + a)\}$, *where* $a = 4$, $b = 5$, $c = 6$, $d = 3$.

(i.) Evaluate the bracket $(2a + b)$: result 13.

(ii.) Evaluate the bracket $(2d + a)$: result 10.

(iii.) Evaluate the bracket $\{3c - 10\}$: result 8.

(iv.) $13 \times 8 = 104$.

We may have brackets within brackets, and these again within other brackets, etc. ; we still work *"outwards" from the inside brackets.*

EXAMPLE (3).—*Evaluate* $(a + b)\{(c^3 + d^3) - 4a[2b - 3(b - c)]\}$; *where* $a = 2$, $b = 8$, $c = 6$, $d = 3$.

(i.) $3(b - c) = 6$

(ii.) $[2b - 6] = 10$; $4a \times 10 = 80$

(iii.) $(c^3 + d^3) = 243$

(iv.) $\{243 - 80\} = 163$

(v.) $(a + b) = 10$

(vi.) $10 \times 163 = 1630$

To place a line over some part of an expression is equivalent to including that part within brackets. When used in this way the line is called a "vinculum."

EXAMPLE (4).—*Evaluate* $3a\{6b - [3c + 2(b - \overline{2c - a})]\}$; *where* $a = 3$, $b = 20$, $c = 2$.

(i.) $\overline{2c - a} = 4 - 3 = 1$

(ii.) $2(b - 1) = 38$

(iii.) $[3c + 38] = 44$

(iv.) $3a\{6b - 44\} = 9\{120 - 44\} = 684$

EXAMPLES.—XXXIV.

If $a = 2$, $b = 3$, $c = 4$, $d = 5$, $e = 1$, evaluate—

1. $a + b\{c - e(d - 3)\}$.

2. $\{(a + b) - c\}d - e$.

3. $(a + b)\{c - e(d - 3)\}$.

4. $\{(a + b) - c\}(d - e)$.

5. $(a^2 + b^2)\{(c^2 - e^2)d - 3(d^2 - b^2)\}$.

6. $2a\{b^2 + 3(c^2 - a^2) - 4d\} + 2b(ac - ed)^2$.

7. $3\{(a^2 + b^2 - bc) + ab(2c - d)^3\}$.

8. $(a + b)^2\{(c + d)^2 - a^2 - b^2\}$.

9. $3ab - \{(a + b)^2 - cd\}$.

10. $a^2 + b^2\{c^2 - e^2(d - 3)^2\}$.

11. $2a + 3b\{4c - [3d - \overline{ab + 5e}]\}$.

12. $(2a + 3b)^2\{4c - 3(d - \overline{ab - 3e})\}$.

13. $3(a + 3b) - \{(4c + 3d) - 2[(3a + b) - 2c]\}$.

14. $\{5a + 2b(3c - 2d)\}\{3a + d[2c - a(b + 1)]\}$.

15. $2\{a + b[c + d(e + 1)^2] + 3c\} - (a + b + c)^2$.

16. $a\{b + c[d - e(d - \overline{b + e})]\}$.

If $a = 2{\cdot}236$, $b = {\cdot}025$, $\pi = 3{\cdot}1416$, calculate correctly to four significant figures—

17. $\pi\{(a + b)^2 - a^2\}$.

18. $\frac{4}{3}\pi\{(a + b)^3 - a^3\}$.

19. $\pi\{(a + b)^2 - b^2\}(2a + b)$.

20. $\frac{4}{3}\pi\{(a + b)^3 - a(a^2 + ab + b^2)\}$.

CHAPTER VII.

ON ALGEBRAICAL ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION.

54. On the Algebraic Meaning of the signs + and − .—Before we can proceed to the subject of this chapter, we must learn the algebraic meaning of the signs + and −. In § 50 we gave the arithmetical meaning; the algebraic meaning is of much wider import.

Whenever we have a set of quantities which can be divided into two groups, such that *any quantity in the one group will neutralize an equal quantity in the other group*, we apply the **sign** + to each quantity in the one group, and the **sign** − to each quantity in the other group. Quantities in the first group are then called "**positive**" quantities; those in the second group, "**negative**" quantities.

The simplest example of such quantities will be gains and losses; if a merchant gains £5 on one transaction, and loses £5 on another, his gain and loss neutralize.

This result would be represented algebraically by the statement $+5-5=0$.

But if the merchant gains £5 on one transaction and loses £7 on another, he has an ultimate loss of £2 on the two transactions.

This result would be represented by the statement $+5-7=-2$; the 2 being preceded by the sign −, because it represents a loss.

Note that in arithmetic $5-7$ would have no meaning, because we cannot subtract 7 from 5.

Now, suppose the various transactions of a day's business gave the following results: loss of £2, loss of £3, gain of £1, loss of £8, gain of £5, gain of £20, loss of £17; his day's work would leave him an ultimate loss of £4.

Representing this algebraically, we write—

$$-2-3+1-8+5+20-17=-4$$

In order, then, to **combine a series of positive and negative quantities**, we must obviously follow this

RULE.—**Find the difference between the sum of the positive quantities and the sum of the negative quantities; prefix the sign + to the result, if the sum of the positive quantities is**

greater than the sum of the negative quantities; prefix the sign − to the result, if the sum of the negative quantities is the greater.

This process of combining a series of positive and negative quantities is called **addition** (although arithmetically it is partly a subtraction); frequently we write the quantities in a column, as in arithmetical addition.

EXAMPLES (1).—

$$\begin{array}{r} -4 \\ +5 \\ -2 \\ +1 \\ \hline 0 \end{array} \qquad \begin{array}{r} +5 \\ -3 \\ -1 \\ +2 \\ \hline +3 \end{array} \qquad \begin{array}{r} -20 \\ +37 \\ -105 \\ \hline -88 \end{array}$$

For the future we shall use the term "addition" in this algebraic sense.

The sign + is often omitted before the first quantity in a line, or before any quantity in a column. The sign − must, of course, never be omitted.

EXAMPLES (2).—*If* $a = 1$, $b = 2$, $c = 5$, $d = 10$, *evaluate the following:* (i.) $a - b + 2c - 3d$. (ii.) $2ab - 3cd + 5ad$.

(i.) $a - b + 2c - 3d = 1 - 2 + 10 - 30 = 11 - 32 = -21$

(ii.) $2ab - 3cd + 5ad = 4 - 150 + 50 = 54 - 150 = -96$

EXAMPLE (3).—*A log floating on a tidal river is observed to be exactly beneath a bridge at high tide; if the successive turns of the tide carry it as follows:* 2 *miles down stream*, $1\frac{3}{4}$ *miles up stream*, $1\frac{1}{2}$ *miles down stream*, $1\frac{3}{4}$ *miles up stream*, $2\frac{1}{2}$ *miles down stream*, 3 *miles up stream*, 1 *mile down stream; represent algebraically the calculation which determines the ultimate position of the log.*

These quantities carry out the essential condition that a quantity of the one kind neutralizes an equal quantity of the other kind. We may therefore represent one group (*whichever group we choose*) as *positive* quantities, and the other group as *negative* quantities. Suppose we reckon distances *down stream* as *positive:* we then obtain—

$$+2 - 1\tfrac{3}{4} + 1\tfrac{1}{2} - 1\tfrac{3}{4} + 2\tfrac{1}{2} - 3 + 1 = 7 - 6\tfrac{1}{2} = +\tfrac{1}{2}$$

Hence the log is ultimately half a mile from the bridge *down stream.*

EXAMPLES.—XXXV.

1. Evaluate $-3 + 7 - 2 + 1 - 4 - 1 - 2 + 10$.

2. Evaluate $-3 - 5 - 8 + 2 + 20 - 10 + 3$.

If $a = 3, b = 2, c = 8, d = 1, e = 0$, evaluate—

3. $2a - 3b - 3c + 5d$.

4. $2ab + 3cd - 2ae + 8bc - 9cd$.

5. $3a^2 - 4b^2 + 5c^2 - 6d^2 - 3abc - 5cde$.

6. Add together the following quantities: -35, $-27{\cdot}5$, $+18{\cdot}3$, $-7{\cdot}2$, $+50$.

7. Add together $-2\frac{3}{4}$, $+2\frac{1}{2}$, -8, $-7{\cdot}3$, $+10\ 5$.

8. The level of the water in a reservoir undergoes the following changes in succession: a rise of 3 inches, a fall of 2·25 inches, a fall of ·3 inch, a rise of ·45 inch, a rise of 2·2 inches, a fall of 1·8 inches. Represent algebraically the calculation which determines its ultimate rise or fall.

9. A cattle-merchant's books show the following transactions in a week: sold, 20 cows at £7 each; sold, 100 sheep at £2 each; wages paid to drovers, £10; bought, 15 cows at £5 each; bought, 50 sheep at £2 each; sold, 20 cows at £8 each; spent in food for cattle, £7. Represent algebraically the calculation which would determine how much more or less money he has in hand at the end of the week than he had at the beginning.

55. Definition.—The coefficient of a term is the numerical multiplier of the term with the sign + or − prefixed; *e.g.* the coefficient of $3ab$ is $+3$, the coefficient of $-\frac{4}{5}a$ is $-\frac{4}{5}$.

If there is no coefficient expressed, the coefficient must be reckoned as 1; *e.g.* a means $1 \times a$, bc^2 means $1 \times bc^2$.

RULE.—**If we have several terms which all have the same letter or letters (and the same powers of the letter or letters), we may add them together, by adding their coefficients, and retaining the letter or letters unaltered.**

EXAMPLE (1).—*Add* $5a$, $-2a$, $-2a$, $6a$.

$$\text{Adding the coefficients:}\quad \begin{array}{r} 5 \\ -2 \\ -2 \\ 6 \\ \hline 7 \end{array} \qquad \text{thus:}\quad \begin{array}{r} 5a \\ -2a \\ -2a \\ 6a \\ \hline 7a \end{array}$$

EXAMPLES (2).—*Add* $-10a^2bc$, $+5a^2bc$, $+3a^2bc$, $-17a^2bc$, $+8a^2bc$.

$$\text{Adding the coefficients:}\quad \begin{array}{r} -10 \\ +5 \\ +3 \\ -17 \\ +8 \\ \hline -11 \end{array} \qquad \text{thus:}\quad \begin{array}{r} -10a^2bc \\ +5a^2bc \\ +3a^2bc \\ -17a^2bc \\ +8a^2bc \\ \hline -11a^2bc \end{array}$$

EXPLANATION.—In Example (1), whatever quantity a may represent, the sum of $5a$ and $6a$ means the sum of 5 times this quantity and 6 times this quantity; the result is 11 times this quantity, which is represented by $11a$; similarly, the sum of the two quantities $-2a$ and $-2a$ is $-4a$; finally, if we subtract 4 times a from 11 times a, the result is 7 times a, which is represented by $7a$.

In Example (2), adding $5a^2bc$ to $3a^2bc$ means adding 5 times a^2bc to 3 times a^2bc, and whatever a^2bc may represent this gives 8 times a^2bc, *i.e.* $8a^2bc$; etc. Notice particularly that the *argument fails unless the terms all have the same letter or letters;* we cannot simplify $5a + 3b$ unless we know the numerical values of a and b; but $5a + 3a$ is always equal to $8a$. Also we cannot simplify $5a^2 + 3a$, because a^2 and a represent different quantities.

EXAMPLE (3).—Simplify $5ax^2 - 2ax^2 + 10ax^2 - 7ax^2 - ax^2$.

Adding the coefficients:

$$\begin{array}{r} 5 \\ -\ 2 \\ +\ 10 \\ -\ 7 \\ -\ 1 \\ \hline +\ 5 \end{array} \qquad \text{thus:} \qquad \begin{array}{r} 5ax^2 \\ -\ 2ax^2 \\ +\ 10ax^2 \\ -\ 7ax^2 \\ -\ ax^2 \\ \hline +\ 5ax^2 \end{array}$$

Notice that the coefficient of $-ax^2$ is -1.

In all these cases the student should carry out the process mentally, and merely write down the result.

EXAMPLES.—XXXVI.

Add together—

1. $8x, -3x, -4x, 2x, 7x, -5x$.
2. $20ab, 10ab, -17ab, -18ab, ab$.
3. $-a^2bc, -20a^2bc, +15a^2bc, +120a^2bc$.
4. $7x^3, -10x^3, 3x^3, -5x^3, -6x^3$.
5. $2abcdxy, 20abcdxy, -50abcdxy, 27abcdxy$.
6. $-11x^2y^2z^2, +21x^2y^2z^2, -15x^2y^2z^2, x^2y^2z^2$.

Simplify—

7. $2ab - 3ab + 4ab - 6ab + 10ab$.
8. $20b^2c^3 - 25b^2c^3 + 70b^2c^3 + 2b^2c^3 - 80b^2c^3$.
9. $a^3b^3 + 5a^3b^3 - 7a^3b^3 + 9a^3b^3 - 10a^3b^3$.
10. $7l^2m^3 - 20l^2m^3 + 3l^2m^3 - 15l^2m^3 + 45l^2m^3$.

56. Addition.—Algebraic expressions may be added together by the following

RULE.—**Place the quantities under one another, arranging all terms which have the same letter or letters, and the same powers of the letter or letters, in a column. Each column can then be added separately.**

EXAMPLE (1).—*Add together* $2a + 3b - c$; $3a - b + 5c$; $2b + 6c - 7a$.

$$\begin{array}{rrr} 2a & +\ 3b & -\ c \\ 3a & -\ b & +\ 5c \\ -\ 7a & +\ 2b & +\ 6c \\ \hline -\ 2a & +\ 4b & +\ 10c \end{array}$$

Note that we have changed the order in the last line, which is permissible, since the Commutative Law applies.

EXAMPLE (2).—*Add together* $3x^2 - 2x + 5$; $5x^2 + 10$; $22x - 10 - 3x^3$; $20x^2 - 20x + 4 - 8x^3 + 4x^4$.

$$\begin{array}{rrrrr} & & 3x^2 & -\ 2x & +\ 5 \\ & & 5x^2 & & +\ 10 \\ & -\ 3x^3 & & +\ 22x & -\ 10 \\ 4x^4 & -\ 8x^3 & +\ 20x^2 & -\ 20x & +\ 4 \\ \hline 4x^4 & -\ 11x^3 & +\ 28x^2 & & +\ 9 \end{array}$$

Note that this may leave gaps in the columns, but this is of no consequence.

It is usual, though not essential, to arrange the different powers of x in "**descending order**," *i.e.* **commencing with the highest power,** followed by the other powers in order, and finishing with the term which does not contain x.

Note also that in the last column the two quantities $+10$ and -10 (which are "equal in magnitude and opposite in sign") neutralize one another, and may therefore be cancelled.

EXAMPLE (3).—*Add together* $\frac{1}{2}a^3 - b^2c + 2abc$; $5b^2c - \frac{1}{5}a^3$; $7bc - \frac{1}{2}abc + \frac{1}{3}a^3$; $-3a^3 - \frac{1}{2}bc + 5abc$.

$$\begin{array}{llll} \frac{1}{2}a^3 & - b^2c & + 2abc & \\ -\frac{1}{5}a^3 & + 5b^2c & & \\ +\frac{1}{3}a^3 & & -\frac{1}{2}abc & + 7bc \\ -3a^3 & & + 5abc & -\frac{1}{2}bc \\ \hline -\frac{71}{30}a^3 & + 4b^2c & + \frac{13}{2}abc & + \frac{13}{2}bc \end{array}$$

Note that the "terms in bc" must be put in a separate column from the "term in b^2c." The order in which the terms are arranged is of little consequence.

57. In the same way, **if an expression contains two or more terms which have the same letter or letters, and the same power of the letter or letters, we may combine these into a single term.**

EXAMPLE.—*Simplify* $3a^2 + 4ab + 5ac - 6ab + 4a^2 - 7bc - 6a^2 + 10ac + 2bc$.

Here we have three "terms in a^2," viz. $3a^2$, $4a^2$, $-6a^2$; when combined, these give a^2.

We have two terms in ab, viz. $4ab$ and $-6ab$; when combined, these give $-2ab$. Proceeding in this way we obtain the result—

$$a^2 - 2ab + 15ac - 5bc$$

EXAMPLES.—XXXVII.

Add together—

1. $2a + 3b - 4c$; $5b - 6c + 7a$; $8c - 9a + 10b$.

2. $3x^2 + 4y^2 - 5z^2$; $2y^2 - 3z^2 + 4x^2$; $6z^2 - 7x^2 + 8y^2$.

3. $20ab + 30ac + 40bc + a^2$; $10ab - 20ac + 30bc - 2a^2$; $4ab - 14ac + 7a^2$; $-8a^2 - 20ac - 30bc$.

4. $2a + 3b - \frac{1}{4}c$; $5b - 6c + \frac{1}{2}a$; $\frac{1}{2}c - \frac{1}{3}a + \frac{1}{4}b$.

5. $\frac{1}{2}a^2 + \frac{1}{3}b^2 - \frac{1}{4}c^2$; $\frac{1}{2}b^2 - \frac{1}{3}c^2 + \frac{1}{4}a^2$; $\frac{1}{2}c^2 - \frac{1}{3}a^2 + \frac{1}{4}b^2$.

6. $\frac{1}{4}abcd + \frac{1}{3}bcd + \frac{1}{2}cd$; $\frac{1}{2}abcd - \frac{1}{4}cd - d$; $\frac{1}{5}bcd - 2cd - 3d$.

7. $\frac{1}{3}x^3 + 2x^2y + y^3$; $\frac{1}{2}x^3 + 3xy^2 - y^3$; $x^3 - \frac{1}{2}x^2y$; $x^2y - \frac{1}{3}xy^2 + \frac{1}{2}y^3$.

8. $\frac{2}{5}a^2 + \frac{3}{5}b^2 - \frac{1}{2}ab$; $\frac{1}{5}a^2 - \frac{2}{5}ab + \frac{1}{2}b^2$; $\frac{3}{10}a^2 + \frac{3}{10}b^2 - \frac{7}{10}ab$.

Simplify—

9. $a^2 + 2b^2 + 3c^2 - 2ab + 3b^2 + 2ac - 5c^2 - 6ab + 8ac$.

10. $3x^3 + 5x^2 - 7x - 8 - 8x - 10x^2 + 11x^3 - 4x + 7x^2$.

11. $4x^3 + 3x^2y - xy^2 - y^3 + \frac{1}{2}xy^2 - \frac{1}{2}x^2y + \frac{1}{3}x^3$.

12. $\frac{1}{2}a + \frac{1}{3}b + \frac{1}{4}c - \frac{1}{5}a - \frac{1}{6}b - \frac{1}{7}c + a - b + c$.

Add—

13. $p + \frac{1}{2}q + \frac{1}{3}r$; $-p + \frac{1}{2}q + \frac{1}{3}r$; $p - \frac{1}{2}q + \frac{1}{3}r$; $p + \frac{1}{2}q - \frac{1}{3}r$.
14. $a^2 + 2ab + b^2$; $b^2 - 2bc + c^2$; $c^2 + 2ca + a^2$.
15. $a^3 + 3a^2b + 3ab^2$; $\frac{1}{2}a^2b + \frac{1}{2}ab^2 - b^3$; $a^3 - \frac{1}{4}b^3 + c^3$.
16. $a^5 + \frac{1}{3}a^4 + \frac{1}{2}a^3$; $a^4 - \frac{1}{3}a^3 - \frac{1}{5}a^2$; $a^2 + a + 5$; $-\frac{1}{4}a^5 + \frac{1}{3}a^3 - a + \frac{2}{3}$.

58. Subtraction.

RULE.—**Change the sign of the quantity which is to be subtracted, and add it to the other quantity.**

N.B.—In adding, we must adhere strictly to the rules of the preceding paragraphs.

EXAMPLE (i.).—*Subtract* -3 *from* $+5$.
Changing the sign of -3, we obtain $+3$; adding $+3$ to $+5$, we obtain $+8$.

EXAMPLE (ii.).—*Subtract* $7abc$ *from* $-3abc$.
Changing the sign of $7abc$, we obtain $-7abc$; adding $-7abc$ to $-3abc$, we obtain $-10abc$.

EXAMPLE (iii.).—*From* $-5a^2$ *subtract* $3a^2$.
Changing the sign of $3a^2$, we obtain $-3a^2$; adding $-3a^2$ to $-5a^2$, we obtain $-8a^2$.

EXAMPLE (iv.).—*From* $-3ab$ subtract $+2bc$.
Changing the sign of $+2bc$, we have $-2bc$; adding $-2bc$ to $-3ab$, we have $-3ab - 2bc$, which cannot be simplified.

EXPLANATION.—The reason for this rule depends entirely on the meaning of the term "subtraction." To subtract a quantity, p, from a quantity, q, means to find that quantity which, when added to p, makes the result equal to q.
The above rule is a mechanical method for discovering this quantity.
Take, for instance, Example (iii.).
We are required to find what quantity must be added to $3a^2$ to give $-5a^2$; but, if to $3a^2$ we add $-3a^2 - 5a^2$, we obviously get $-5a^2$ (since $3a^2$ cancels with $-3a^2$). Thus $-3a^2 - 5a^2$ is the required quantity; *i.e.* $-8a^2$.
Similarly, in Example (i.), if to -3 we add $+3 + 5$, we obtain the required result, $+5$; hence the result of subtracting -3 from $+5$ is $+3 + 5$; *i.e.* $+8$.

EXAMPLES.—XXXVIII.

Subtract—

1. -3 from -5. **2.** -10 from 2. **3.** 8 from -2.
4. 8 from 5. **5.** 7 from 2. **6.** $3{\cdot}5$ from $-3{\cdot}5$.
7. $-2{\cdot}5$ from $-8{\cdot}9$. **8.** 21 from -10. **9.** $2{\cdot}1$ from -10.
10. $-2{\cdot}5$ from $-1{\cdot}2$. **11.** -35 from -17. **12.** -2 from $+7$.
13. $-1{\cdot}5$ from $6{\cdot}3$. **14.** $-5a^2b$ from $2a^2b$. **15.** $8abc$ from $-2abc$.
16. $10x^3$ from $3x^3$. **17.** $2{\cdot}5x^2y$ from $-1{\cdot}5x^2y$.
18. $-3{\cdot}8xyz$ from $-20xyz$. **19.** $2x^3y^3z^3$ from $-12x^3y^3z^3$.
20. $-3{\cdot}8x^2$ from $-1{\cdot}5x^2$. **21.** $3ab$ from $2bc$.
22. $6abc$ from $-3ab^2$. **23.** $-3a^2$ from $+5a^3$.
24. $-2bc$ from $-5b^2c$. **25.** $-6pq$ from $-10pqr$.

59. The same rule, applied term by term, enables us to subtract one quantity from another, when one or both quantities contain more than one term.

EXAMPLE (i.).—*From* $3a^2 - 5ab + 7b^2$ *subtract* $a^2 + 3ab + 10b^2$.

Changing the sign of each term in the quantity to be subtracted, we obtain—

$$-a^2 - 3ab - 10b^2$$

Adding this to the other quantity—

$$\begin{array}{r} 3a^2 - 5ab + 7b^2 \\ -\ a^2 - 3ab - 10b^2 \\ \hline 2a^2 - 8ab - 3b^2 \end{array}$$

EXAMPLE (ii.).—*From* $3abc - a^3 - b^3 - c^3$ subtract $a^3 - 3a^2b - 3abc - b^3 - c^3$.

Changing the sign of each term in the quantity to be subtracted, we have—

$$-a^3 + 3a^2b + 3abc + b^3 + c^3$$

Place this under the other quantity, with *corresponding terms in the same column*, and add—

$$\begin{array}{l} \quad 3abc - a^3 - b^3 - c^3 \\ +\ 3abc - a^3 + b^3 + c^3 + 3a^2b \\ \hline \quad 6abc - 2a^3 \qquad\qquad + 3a^2b \end{array}$$

It is more usual *not* to write down the one quantity with its signs changed, but to change the signs *mentally*, *after* placing the quantity to be subtracted under the other quantity; but this is a matter of little importance.

EXAMPLE (iii.).—*Subtract* $3a^2 - 5b^2 + 6c^2 - 10ab$ *from* $2a^2 + 7b^2 - 3ab + 5bc$.

$$\begin{array}{rrrrr} 2a^2 & +\ 7b^2 & -\ 3ab & +\ 5bc & \\ 3a^2 & -\ 5b^2 & -\ 10ab & & +\ 6c^2 \\ \hline -\ a^2 & +\ 12b^2 & +\ 7ab & +\ 5bc & -\ 6c^2 \end{array}$$

The mental process is then as follows :—

In the first column :	$-\ 3a^2 + 2a^2$	$= -\ a^2$
,, second column :	$+\ 5b^2 + 7b^2$	$= +\ 12b^2$
,, third column :	$+\ 10ab - 3ab$	$= +\ 7ab$
,, fourth column :	$+\ 0 + 5bc$	$= +\ 5bc$
,, fifth column	$-\ 6c^2 + 0$	$= -\ 6c^2$

EXAMPLES.—XXXIX.

Subtract—

1. $3a^2 - 3ab + 3b^2$ from $5a^2 + 5ab - 10b^2$.
2. $3x^2 + 2x^2y - 2xy^2 - 5y^3$ from $5x^3 - 3x^2y - 8xy^2 + 10y^3$.
3. $10p^2 + 12pq + 13pr - 25q^2$ from $15p^2 - 10pq + 10pr - q^2$.
4. $5x^2 + 5xy + 6y^2 - 10yz + z^2$ from $3x^2 + 10y^2 - 10xz + 5yz + 3z^2$.
5. $\frac{1}{4}x^3 + 3x + \frac{1}{7}$ from $\frac{1}{2}x^2 + 7x + \frac{1}{15}$.

6. $\frac{2}{3}x^4 + \frac{1}{2}x^2 + \frac{1}{6}$ from $\frac{5}{6}x^4 - x^3 - 2$.

7. $\cdot 24a + \cdot 36b - \cdot 8c$ from $\cdot 36a - \cdot 24b + \cdot 3c$.

8. $3a - \cdot 2b + \cdot 5$ from $2\cdot 6a + \cdot 8b - \cdot 3$.

9. $\cdot 4ab + \cdot 5bc - \cdot 6ca$ from $ab + bc + ca$.

10. $3ab - 4ac + 5c^2$ from $a^2 - 3ac - 2c^2$.

11. What quantity must be added to $4a^2 - 4bc + 4c^2$ in order that the sum may be $2a^2 + 4ac + 3c^2$?

12. What quantity must be added to $\frac{1}{2}a^2 - \frac{1}{3}b^2 + \frac{1}{5}c^2$ in order that the sum may be $\frac{1}{3}a^2 + \frac{1}{2}b^2 + 3bc - c^2$?

13. What quantity must be subtracted from $3x^2 - 2xy - 5y^2 + z^2$ in order that the difference may be $2xy + 5y^2 - 2z^2$?

14. What quantity must be subtracted from $x^2 + y^2 - z^2 - xy$ in order that the difference may be $\cdot 1x^2 - \cdot 2y^2 + \cdot 3z^2 + yz$?

15. From $3a - 2b + c$ subtract $2d - 3e + 5f$.

16. The sum of two quantities is $\cdot 4x^2 + \cdot 5y^2 - \cdot 7z^2$; one of the quantities is $\cdot 5x^2 - \cdot 4y^2 - \cdot 8z^2$: find the other.

17. The difference between two quantities is $3x^2 - 4y^2 + 3xz + \frac{1}{2}yz + 2z^2$; one of them is $2x^2 + 2y^2 - \frac{1}{2}yz + 5z^2$: find the other.

18. What quantity must be added to $\cdot 234x^2 - \cdot 345y^2 + \cdot 678z$ in order that the sum may be $\cdot 432x^2 + \cdot 543y^2 - \cdot 876z$?

19. What quantity must be subtracted from $\cdot 235a^2 + \cdot 871$ in order that the difference may be $\cdot 268a^2 - \cdot 23a$?

20. The sum of two quantities is $\cdot 1x^2 + \cdot 2y^2 - \cdot 3z^2 - \cdot 4xy + \cdot 5yz$; one of them is $\cdot 3y^2 + \cdot 2z^2 - \cdot 345xy + 1\cdot 2yz + \cdot 346xz$: find the other.

60. On the Multiplication of Single Terms.

RULE.—**To multiply together two terms which contain *different* letters, we write down the product of the coefficients, followed by all the letters of both terms, in any order, with unchanged indices.**

EXAMPLES.—(i.) $3a^2 \times 4b^3 = 12\,a^2b^3$

(ii.) $5xy \times 7z^2 = 35xyz^2$

(iii.) $2abe \times \frac{1}{4}cd = \frac{1}{2}abcde$

EXPLANATION.—In Example (ii.): From § 47 we know that $5xy \times 7z^2$ means $5 \times x \times y$ multiplied by $7 \times z^2$, which gives $5 \times x \times y \times 7 \times z^2$. But since a *series of factors* may be multiplied *in any order*, this is equivalent to $5 \times 7 \times x \times y \times z^2$, or $35 \times x \times y \times z^2$. The multiplication signs may be omitted, (see § 47) and the answer written $35xyz^2$.

Remember that if no coefficient is expressed, the coefficient is 1; thus—

$$3a^2b \times c^2d = 3 \times 1 \times a^2 \times b \times c^2 \times d = 3a^2bc^2d$$

RULE.—**The product of two powers of the same letter is that power of the same letter whose index is the sum of their two indices.**

EXAMPLES.—(iv.) $a^3 \times a^4 = a^7$

(v.) $a^2 \times a = a^2 \times a^1 = a^3$

(vi.) $x^2 \times x^7 = x^9$

In Example (ii.) note that the index of a is 1.

EXPLANATION.—In Example (iv.), a^3 means $a \times a \times a$ (see § 33); a^4 means $a \times a \times a \times a$; hence their product is $a \times a \times a \times a \times a \times a \times a$, which is a^7.

In Example (ii.), a^2 means $a \times a$.

$$\therefore a^2 \times a = a \times a \times a = a^3$$

This rule is usually quoted in the shorter form, "To multiply powers of the same letter, add the indices."

By combining these two rules, we may multiply together any terms.

Rule for multiplying any terms—

First, multiply the coefficients.

Second, multiply any powers of the same letter which may occur, by adding their indices.

Third, write down the other letters which occur in both terms.

EXAMPLES.—(vii.) $3a^2b \times 5a^3c = 3 \times 5 \times a^2 \times a^3 \times b \times c$
$= 15a^5bc$

(viii.) $2a^3b^2 \times 5a^2b^5c^3 = 2 \times 5 \times a^3 \times a^2 \times b^2 \times b^5 \times c^3$
$= 10a^5b^7c^3$

(xi.)
$3a^3c^2 \times b^5cd \times 4a^2b^2c^3 = 3 \times 1 \times 4 \times a^3 \times a^2 \times b^5 \times b^2 \times c^2 \times c \times c^3 \times d$
$= 12a^5b^7c^6d.$

(x.) $\frac{1}{2}x^5 \times \frac{2}{5}x^7 = \frac{1}{2} \times \frac{2}{5} \times x^5 \times x^7$
$= \frac{1}{5}x^{12}$

EXAMPLES.—XL.

Evaluate—

1. $3x^2 \times 2x^3$; $27x \times 5x^4$; $2x^5 \times x$; $30x^2 \times 2x^{10}$.
2. $3a^2b \times 2ab^3$; $10x^2y \times 3xy^5$; $5x^2y^3 \times \frac{1}{10}x^5y^7$.
3. $4ab \times 7cd$; $2xy \times 15zw$; $12abd \times 12cef$.
4. $\frac{2}{3}ab \times \frac{1}{6}ace$; $3a^2e \times 2a^3ce$; $\frac{5}{6}ad^3 \times \frac{3}{20}b^2d^4e$.
5. $2a^2b^3c \times \frac{3}{10}ab^4c^5$; $6a^5bcd^2 \times 10a^3c^5$.
6. $\frac{3}{5}lm^2n \times \frac{3}{8}m^3$; $\frac{20}{21}p^2q \times \frac{3}{5}q^2r$; $\frac{1}{7}xy \times \frac{21}{4}y^7z$.
7. $\frac{2}{3}a^3be \times \frac{3}{8}a^2cd^3$; $\frac{1}{5}l^2m \times \frac{2}{5}k^2m^3n$.
8. $\cdot 4abc \times \cdot 5abc$; $\cdot 6lmn \times \cdot 8ln$; $\cdot 2xy \times \cdot 51yz$.
9. $\cdot 21a^3b^3 \times \cdot 7a^2b^4c$; $\cdot 3xy^2 \times \cdot 23y^3z^5$.
10. $3xy \times 5ax^2z \times 10a^2bxz^2$; $5x^2y^3 \times 7a^3x^2 \times 10bx^3z^2$.
11. $2ab \times 3b^2c^2 \times 4a^3c^3 \times 5a^4b^4 \times 6b^5c^5$.
12. $\cdot 3abx \times \cdot 4a^2c^2x^2y^2 \times \cdot 5b^3c^3y^3z^3 \times \cdot 6a^4b^4x^4z^4$.

61. To multiply any expression by a Single Positive Term.

RULE.—**Multiply each term of the expression separately, and bring down the signs as they occur.**

EXAMPLE (1).—*Multiply* $3a + 2b - 4c$ by $5a$.

$$\begin{array}{l} 3a + 2b - 4c \\ 5a \\ \hline 15a^2 + 10ab - 20ac \end{array}$$

Notice that the multiplication of each term is performed in accordance with the rules of the preceding paragraph.

EXAMPLE (2).—*Multiply* $3x^2 - 5xy - 20y^2$ by $5x^3$.

$$\begin{array}{l} 3x^2 - 5xy - 20y^2 \\ 5x^3 \\ \hline 15x^5 - 25x^4y - 100x^3y^2. \end{array}$$

EXPLANATION.—We know from Arithmetic that 6 times $(8 + 3 - 4)$ is the same as 6 times $8 + 6$ times $3 - 6$ times 4. This principle is, of course, true for any quantity. Thus in Example (1) it follows that $5a$ times $(3a+2b-4c)$ is the same as $5a$ times $3a + 5a$ times $2b - 5a$ times $4c$.

EXAMPLES.—XLI.

Multiply—

1. $3a^2 - 4ab + 5b^2$ by 7.
2. $2a^3 + 4a^2b - 5ab^2 + 6b^3$ by 12.
3. $4x^3 + 3x^2 - 4x - 5$ by $4x^2$.
4. $10a^2 + 5b^2 + 3c^2 - 15ab + 10bc - 12ac$ by $20ab$.
5. $10x^4 - 7x^3y + 8x^2y^2 + 5xy^3 - 12y^4$ by $5x^3y^3$.
6. $\cdot 2a + \cdot 3b - \cdot 4c + \cdot 5d - \cdot 6e$ by $20abcde$.
7. $\cdot 2l^2 + \cdot 3m^2 + \cdot 4n^2 - \cdot 5mn - \cdot 6nl - \cdot 7lm$ by $\cdot 8l^2m^2$.
8. $-20p^2 - 10q^2 - 5r^2 + 3px + 4qx + 5rx$ by $\cdot 6pqx^2$.
9. $-x^2 - 2y^2 + 3z^2 + 2yz - 3zx + 4xy$ by $\cdot 5x^2y^2z^2$.

62. To multiply any two Algebraical Expressions.—The method is merely an obvious extension of the last rule. The **multiplicand must be multiplied successively by each term of the multiplier,** and the results **added**, according to the ordinary rules of algebraic addition. It is, of course, an advantage to arrange terms which can be added together in the same column; *i.e.* terms which contain the same letters, to the same powers, should be placed in the same column.

EXAMPLE (2).—*Multiply* $3a^2 + 4ab - 5b^2$ *by* $4a^2 + 3ab + 2b^2$.

$$\begin{array}{lllll} 3a^2 & + 4ab & - 5b^2 & & \\ 4a^2 & + 3ab & + 2b^2 & & \\ \hline 12a^4 & + 16a^3b & - 20a^2b^2 & & \\ & + 9a^3b & + 12a^2b^2 & - 15ab^3 & \\ & & + 6a^2b^2 & + 8ab^3 & - 10b^4 \\ \hline 12a^4 & + 25a^3b & - 2a^2b^2 & - 7ab^3 & - 10b^4 \end{array}$$

In this example we first multiply the top line by $4a^2$, then by $3ab$, then by $2b^2$. The results of these multiplications are then added.

There is, however, an important case which we have still to consider. Suppose that in this last example the third term of the multiplier had been $-2b^2$ instead of $+2b^2$. It would then be necessary to multiply the top line by $2b^2$ and *subtract** the result from the rest. The easiest method of performing this subtraction is to *change the signs* in this line of multiplication, and then add it in with the rest (compare §§ 58, 59).

* For example, to multiply a quantity by $(10 - 7)$ is obviously to multiply it by 3. But we shall obtain the correct result if we *subtract* 7 times the quantity from 10 times the quantity.

We thus obtain the following

RULE.—**When multiplying by a positive term, keep the signs of the multiplicand unchanged; when multiplying by a negative term, alter each sign of the multiplicand.**

EXAMPLE (2).—*Multiply* $3a^2 - 4ab + 7b^2$ *by* $2a^2 - 5ab + 2b^2$.

$$\begin{array}{rrrrr}
3a^2 & - 4ab & + 7b^2 & & \\
2a^2 & - 5ab & + 2b^2 & & \\
\hline
6a^4 & - 8a^3b & + 14a^2b^2 & & \\
 & - 15a^3b & + 20a^2b^2 & - 35ab^3 & \\
 & & + 6a^2b^2 & - 8ab^3 & + 14b^4 \\
\hline
6a^4 & - 23a^3b & + 40a^2b^2 & - 43ab^3 & + 14b^4
\end{array}$$

When multiplying the top line by $-5ab$, we multiply by $5ab$ and change all the signs as we put the results down.

63. The above rule for the signs in multiplication is usually given in a different form.

Rule of signs in multiplication:

I. The product of two "like" signs gives a positive result. That is to say, the product of two positive quantities is positive, and also the product of two negative quantities is positive. *E.g.* in Example 2, $+7b^2$ multiplied by $+2a^2$ gives $+14a^2b^2$; also $-4ab$ multiplied by $-5ab$ gives $+20a^2b^2$.

II. The product of the two "unlike" signs gives a negative result. *E.g.* in Example (2), $+7b^2$ multiplied by $-5ab$ gives $-35ab^3$; and $-4ab$ multiplied by $+2b^2$ gives $-8ab^3$.

This form of the rule has the advantage of being concise, and very easy of application.

EXAMPLE (3).—*Multiply* $3x^3 - 2x^2 + 5x - 7$ *by* $3x^3 + 2x^2 - 5x - 7$.

$$\begin{array}{rrrrrrr}
3x^3 & - 2x^2 & + 5x & - 7 & & & \\
3x^3 & + 2x^2 & - 5x & - 7 & & & \\
\hline
9x^6 & - 6x^5 & + 15x^4 & - 21x^3 & & & \\
 & + 6x^5 & - 4x^4 & + 10x^3 & - 14x^2 & & \\
 & & - 15x^4 & + 10x^3 & - 25x^2 & + 35x & \\
 & & & - 21x^3 & + 14x^2 & - 35x & + 49 \\
\hline
9x^6 & & - 4x^4 & - 22x^3 & - 25x^2 & & + 49
\end{array}$$

EXAMPLE (4).—*Multiply* $\frac{1}{4}a^2 + \frac{1}{9}b^2 + 4 + \frac{1}{6}ab - a + \frac{2}{3}b$ *by* $\frac{1}{2}a - \frac{1}{3}b + 2$.

$$\begin{array}{lllllllll}
\frac{1}{4}a^2 + \frac{1}{9}b^2 + 4 + \frac{1}{6}ab - a + \frac{2}{3}b & & & & & & & & \\
\frac{1}{2}a - \frac{1}{3}b + 2 & & & & & & & & \\
\hline
\frac{1}{8}a^3 + \frac{1}{18}ab^2 & + 2a & + \frac{1}{12}a^2b & - \frac{1}{2}a^2 & + \frac{1}{3}ab & & & & \\
\quad - \frac{1}{18}ab^2 & & - \frac{1}{12}a^2b & & + \frac{1}{3}ab & - \frac{1}{27}b^3 & - \frac{4}{3}b & - \frac{2}{9}b^2 & \\
 & - 2a & & + \frac{1}{2}a^2 & + \frac{1}{3}ab & & + \frac{4}{3}b & + \frac{2}{9}b^2 & + 8 \\
\hline
\frac{1}{8}a^3 & & & & + ab & - \frac{1}{27}b^3 & & & + 8
\end{array}$$

Note the entire change of order in the terms in the second and third lines of multiplication. We are, of course, merely following the rule that terms which can be added are placed in the same column.

EXAMPLES.—XLII.

Multiply—

1. $a^2 - 2ab + b^2$ by $a - b$.
2. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
3. $a^3 + 5a^2b - 10ab^2 - 5b^2$ by $a^2 - 5ab + 10b^2$.
4. $16x^4 + 8x^3 + 4x^2 + 2x + 1$ by $2x - 1$.
5. $81x^4 - 27x^3 + 9x^2 - 3x + 1$ by $3x + 1$.
6. $7x^2 + 10xy - 20y^2$ by $10x^2 + 7xy - 20y^2$.
7. $8x^3 + 12x^2y + 6xy^2 + y^3$ by $8x^3 - 12x^2y + 6xy^2 - y^3$.
8. $\frac{1}{2}a + \frac{1}{3}b - c$ by $\frac{1}{2}a - \frac{1}{3}b + c$.
9. $\frac{1}{3}a + \frac{1}{4}b + \frac{1}{2}c - \frac{1}{5}$ by $\frac{1}{3}a - \frac{1}{4}b - \frac{1}{2}c - \frac{1}{5}$.
10. $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{1}{4}z^2 - \frac{1}{5}xy$ by $\frac{1}{2}x^2 - \frac{1}{3}y^2 + \frac{1}{4}z^2 + \frac{1}{5}xy$.
11. $\cdot3x + \cdot4y + \cdot5z$ by $\cdot2x + \cdot3y + \cdot4z$.
12. $\cdot12x^3 + \cdot23x^2 + \cdot34x - \cdot45$ by $\cdot1x^2 - \cdot2x - \cdot3$.
13. $\cdot001x^3 - \cdot01x^2 + \cdot1x - 1$ by $\cdot3x + 3$.
14. $\cdot0027x^4 + \cdot009x^3 + \cdot03x^2 + \cdot1x$ by $\cdot9x - 3$.
15. $\frac{1}{2}x^4 - \frac{1}{3}x^3 + \frac{1}{4}x^2 - \frac{1}{5}x + \frac{1}{6}$ by $\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{4}$.
16. $3a^2 + 3ab + 4ac$ by $3ab - 3b^2 - 4bc$.
17. $6a^2b + 9abc - 15ab^2$ by $2ac - 3c^2 - 5bc$.
18. $x^2 + 2xy + 4y^2$ by $x^2 - 2xy + 4y^2$.
19. $x^4 + \cdot3x^2y^2 + \cdot09y^4$ by $x^4 - \cdot3x^2y^2 + \cdot09y^4$.
20. $b^2 + 2bc + c^2 - a^2$ by $a^2 - b^2 + 2bc - c^2$.
21. $\cdot04x^2 + \cdot12xy + \cdot09y^2 - z^2$ by $z^2 - \cdot04x^2 + \cdot12xy - \cdot09y^2$.
22. $a^2b - a^2c + b^2c - b^2a + c^2a - c^2b$ by $a + b + c$.
23. $2a^2b - 3a^2c + 12b^2c - 4b^2a + 9c^2a - 18c^2b$ by $a + 2b + 3c$.
24. $- 2a^2b + 3a^2c - 12b^2c - 4b^2a + 9c^2a + 18c^2b$ by $a - 2b - 3c$.
25. $a^2b - a^2c + b^2c - b^2a + c^2a - c^2b$ by $bc + ca + ab + a^2 + b^2 + c^2$.
26. $\frac{1}{2}a^2b - \frac{1}{3}a^2c + \frac{1}{12}b^2c - \frac{1}{4}b^2a + \frac{1}{9}c^2a - \frac{1}{18}c^2b$ by $a + \frac{1}{2}b + \frac{1}{3}c$.
27. $\frac{1}{2}a^2b - \frac{1}{3}a^2c + \frac{1}{12}b^2c - \frac{1}{4}b^2a + \frac{1}{9}c^2a - \frac{1}{18}c^2b$ by $\frac{1}{6}bc + \frac{1}{3}ca + \frac{1}{2}ab + a^2 + \frac{1}{4}b^2 + \frac{1}{9}c^2$.
28. $a^2 + b^2 + c^2 - bc + ca + ab$ by $a - b - c$.
29. $x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 - \frac{1}{6}yz + \frac{1}{3}zx + \frac{1}{2}xy$ by $x - \frac{1}{2}y - \frac{1}{3}z$.
30. $\cdot09l^2 + \cdot16m^2 + \cdot25n^2 - \cdot2mn + \cdot15nl + \cdot12lm$ by $\cdot3l - \cdot4m - \cdot5n$.

64. On dividing a Single Term by a Single Term.—The object of division is merely to find that quantity which, when multiplied by the divisor, gives a result equal to the dividend; the rules of division are therefore derived from the rules of multiplication, and will need but little explanation.

We know that $a^5 \times a^3 = a^8$ (see § 60); it follows that $a^8 \div a^5 = a^3$. Hence the

RULE.—**To divide powers of the same letter, subtract the index of the divisor from the index of the dividend.**

EXAMPLES.—

$$x^{20} \div x^{13} = x^7$$
$$12x^{13} \div 2x^5 = 6x^8$$

We know that $ab^3 \times cd^2 = ab^3cd^2$; it follows that $ab^3cd^2 \div ab^3 = cd^2$. Hence the

RULE.—**Letters which occur to the same power in both dividend and division, do not appear in the quotient (a and b^3); letters which occur in the dividend, but not in the divisor, occur unaltered in the quotient (c and d^2).**

EXAMPLES.—

$$ab^2xy^3 \div ay^3 = b^2x$$
$$12a^3b^2c^2d \div 6a^3d = 2b^2c^2$$

The following examples combine the two rules :—

$$15a^3b^2c \div 5ab^2 = 3a^2c$$
$$(\text{for } a^3 \div a = a^2)$$
$$20a^3x^5y^2z \div 10a^2x^2y^2 = 2ax^3z$$
$$(\text{for } a^3 \div a^2 = a\ ;\ x^5 \div x^2 = x^3).$$

A very good method is to *use the ordinary arithmetical rules of cancelling, combined with the law for the division of powers.* The last two examples are then worked as follows :—

$$15a^3b^2c \div 5ab^2 = \frac{\overset{3}{\cancel{15}} \times \overset{a^2}{\cancel{a^3}} \times \cancel{b^2} \times c}{\cancel{5} \times \cancel{a} \times \cancel{b^2}} = 3a^2c$$

$$20a^3x^5y^2z \div 10a^2x^2y^2 = \frac{\overset{2}{\cancel{20}} \times \overset{a}{\cancel{a^3}} \times \overset{x^3}{\cancel{x^5}} \times \cancel{y^2} \times z}{\cancel{10} \times \cancel{a^2} \times \cancel{x^2} \times \cancel{y^2}} = 2ax^3z$$

65. The Rule of Signs in Division.—This is easily derived as follows from the rule of signs in multiplication :—

$+ 5$ multiplied by $+ 3$ gives $+ 15$; $\therefore + 15$ divided by $+ 3$ gives $+ 5$
$+ 5$ multiplied by $- 3$ gives $- 15$; $\therefore - 15$ divided by $- 3$ gives $+ 5$
$- 5$ multiplied by $+ 3$ gives $- 15$; $\therefore - 15$ divided by $+ 3$ gives $- 5$
$- 5$ multiplied by $- 3$ gives $+ 15$; $\therefore + 15$ divided by $- 3$ gives $- 5$

It is easily seen that these rules can be summed up *in the same way as the rules for multiplication* in § 63—

RULES.—I. **In division "like signs" give** $+$; *e.g.*

$$(+ 15) \div (+ 3) = + 5$$
$$(- 15) \div (- 3) = + 5$$

II. **In division "unlike signs" give** $-$; *e.g.*

$$(- 15) \div (+ 3) = - 5$$
$$(+ 15) \div (- 3) = - 5$$

EXAMPLES.—*Divide* $-30x^3y^4z^2$ *by* $+ 10xy^2$.

$$\frac{-\overset{3\ x^2y^2}{\cancel{30x^3y^4z^2}}}{+\cancel{10xy^2}} = -3x^2y^2z^2$$

Divide $-52x^7y^6z^3$ *by* $-13x^7y^2z$.

$$\frac{-\overset{4\ \ y^4z^2}{\cancel{52x^7y^6z^3}}}{-\cancel{13x^7y^2z}} = +4y^4z^2$$

EXAMPLES.—XLIII.

Divide—

1. $30a^3b^2c$ by $15ab^2$. **2.** $25ab^5c^5$ by $5abc^3$. **3.** $26a^5bx^4y$ by $13a^2by$.
4. $24x^4y^5z^6$ by $8xy^2z^3$. **5.** $32a^5x^5y^6z^6$ by $8a^4x^3y^2z$. **6.** $22abc^3a^3$ by $11ac$.
7. $20a^3bc^5$ by $-5abc^3$. **8.** $-20a^5x^6y^6$ by $5a^4x$. **9.** $-36ax^5yz^5$ by $-9ayz^3$.
10. $36ab^3c$ by $-6ab$. **11.** $-27x^8yz^6$ by $-3x^4z^3$. **12.** $27xyz^3$ by $-9xz$.
13. $-28abcd^5$ by $7acd^3$. **14.** $-21x^3y^3z$ by $-7xyz$. **15.** $\cdot 8a^3b^3$ by $-\cdot 2ab$.
16. $\cdot 16x^3y^4$ by $-\cdot 8y^2$.

66. Short Division.—In dividing an expression by a single term, we divide separately into each term of the expression, using the rules of the preceding paragraph.

EXAMPLE (1).—*Divide* $22a^2b^2 + 28a^5bc - 6a^3b^2$ *by* $2a^2b$.

$$2a^2b\overline{)22a^2b^2 + 28a^5bc - 6a^3b^2}$$
$$11b \quad + 14a^3c \quad - 3ab$$

EXAMPLE (2).—*Divide* $\cdot 25a^3x^3y^3 - 20a^2x^4y^4 - 15ax^2y^2$ *by* $-5ax^2y^2$.

$$-5ax^2y^2\overline{)\cdot 25a^3x^3y^3 - 20a^2x^4y^4 - 15ax^2y^2}$$
$$-\cdot 05a^2xy \quad + 4ax^2y^2 \quad + 3$$

EXAMPLES.—XLIV.

Divide—

1. $12a^4x^3y^4 - 16a^3x^4y^2 + 20a^2x^5y$ by $4a^2x^3y$.
2. $-21x^3y^3z^3 + 18x^4y^4z^4 - 12x^3y^3$ by $3x^2y^3$
3. $15p^2q^2 + 45p^3q^3 - 60p^5q^5r^3$ by $5p^2q^2$.
4. $9p^2qr^2 - 6p^3q^2r^2 + 6p^4q^2r^2$ by $-3p^2qr$.
5. $14x^3y^3z^3 - 21x^4y^5z^6 + 28x^2y^5z^4$ by $7xy^3z$.
6. $12a^5b^4c^3 - 18a^4b^3c - 24a^3b^3c^5$ by $-6a^3b^2c$.
7. $10a^3b^2c - 15a^4b^3cd + 5a^3b^2cde$ by $-5a^2b$.
8. $21p^2q^2r^2 - 14p^3q^3r + 28p^4q^4s$ by $-7p^2q^2$.
9. $2\cdot 1x^3y^3 - 1\cdot 2x^2y^2 + 3xy$ by $\cdot 3xy$.
10. $3\cdot 6x^3y^3z - \cdot 24x^2y^2z^2 - \cdot 6xyz^3$ by $1\cdot 2xyz$.

67. On the correct Order of Terms in Algebraic Expressions.—Up to this point the order of the terms in an algebraical expression has been of little or no importance. But **in the process of long division and of evolution the order of the terms is of the greatest importance.**

If an algebraical expression involves one letter only, we place the *highest power* of that letter first, then the next highest *power*,* and so on. Thus the expression $2x - 3 + 5x^3 - x^2 - 2x^4$ must be arranged as follows: $-2x^4 + 5x^3 - x^2 + 2x - 3$. It is then said to be arranged in "**descending powers of** x." If, however, the expression contains more than one letter, we choose one of the letters (it does not matter which) and determine the order by it as far as possible. But if there

* The magnitude of the *coefficients*, i.e. of the *numerical* factor in the term, has no effect on its position.

are several terms *which contain the same power of this letter*, we arrange them *among themselves* according to the powers of one of the other letters.

EXAMPLE.—*Arrange in correct order:* $3a^2b^2c^2 - 4a^3b + 6a^4c^2 - 7abc^4 + 8a^3b^2c - 10b^4c^2 + 5a^2bc + 6a^3c - 5a^4b^2 + 3a^4bc - a^2c^2 + b^2c^4$.

We will arrange as far as possible in descending powers of a.

The highest power of a which occurs is a^4.

(i.) The following terms contain a^4 :—

$$+ 6a^4c^2, - 5a^4b^2, + 3a^4bc$$

(ii.) The following terms contain a^3 :—

$$- 4a^3b, + 8a^3b^2c, + 6a^3c$$

(iii.) The following terms contain a^2 :—

$$+ 3a^2b^2c^2, + 5a^2bc, - a^2c^2$$

(iv.) The following term contains a :—

$$- 7abc^4$$

(v.) The following terms are without a :—

$$- 10b^4c^2, + b^2c^4.$$

We will arrange each group *among themselves* in descending powers of b; thus in the first group the order will be $-5a^4b^2 + 3a^4bc + 6a^4c^2$, the term without b coming at the end of the group. Proceeding in the same way with the other groups, we obtain ultimately the following order of terms :—

$$\begin{aligned} &- 5a^4b^2 + 3a^4bc + 6a^4c^2 \\ &+ 8a^3b^2c - 4a^3b + 6a^3c \\ &+ 3a^2b^2c^2 + 5a^2bc - a^2c^2 \\ &- 7abc^4 \\ &- 10b^4c^2 + b^2c^4 \end{aligned}$$

Note that in no case has the *numerical coefficient* any effect on the position of the term. This is settled primarily by the index of a; and where two terms have the same index of a, their order is settled by the index of b. It would not be incorrect to settle the order primarily by the index of b, and secondarily by the index of a (or of c), provided we are consistent throughout the operations in hand; but it is usual to work as above.

EXAMPLES.—XLV.

Arrange in descending powers of x—

1. $3x^2 - \cdot 3x + \cdot 07x^3 - \cdot 182x^4 - \cdot 23x^6 + \cdot 5 - x^5$.

2. $5x^3 - 5x^2 + \frac{1}{2}x^5 - \frac{1}{3}x + 5$. **3.** $ax^3 + bx^4 - cx^2 + dx^5 - ex + f$.

Arrange in descending powers of x (and in descending powers of y, where the powers of x are the same)—

4. $x^3 + y^3 + z^3 - 3xyz$. **5.** $x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2$.

6. $x^5 + 3y^5 + 5z^5 + 4x^2y^3 - 5x^3y^2 + 2y^3z^2 - 2y^2z^3 + z^3x^2 - 4x^3z^2$.

7. $4y^4 - 3z^4 - 2xy^3 + 4x^3y - 3xz^3 - 5x^3z$.

8. $ay^5 + by^4z + cy^4x - dy^3x^2 + ey^2z^3 + fyx^4 - gz^5 - hx^5$.

68. Long Division.—This process corresponds very closely to long division in Arithmetic. But it is essential to arrange the terms in correct order, both in the dividend and divisor, and also in the various remainders.

EXAMPLE (1).—*Divide* $15x^4 - 8x^3 + 32x - 15$ *by* $3x^2 - 4x + 5$.

$$
\begin{array}{l}
3x^2 - 4x + 5)15x^4 - 8x^3 + 32x - 15(5x^2 + 4x - 3 \\
\qquad\qquad\quad 15x^4 - 20x^3 + 25x^2 \\
\hline
\qquad\qquad\qquad\quad 12x^3 - 25x^2 + 32x \\
\qquad\qquad\qquad\quad 12x^3 - 16x^2 + 20x \\
\hline
\qquad\qquad\qquad\qquad - 9x^3 + 12x - 15 \\
\qquad\qquad\qquad\qquad - 9x^2 + 12x - 15
\end{array}
$$

We first divide $3x^2$ into $15x^4$, and place the result, viz. $5x^2$, in the quotient. We then multiply the divisor by $5x^2$, and subtract it from the dividend, bringing down the next term.

We next divide $3x^2$ into $12x^3$, and place the result, viz. $4x$, in the quotient. Then we multiply the divisor by $4x$, and subtract it from the last remainder; and so on. Each subtraction is performed by the method of § 59.

The terms from the dividend are brought down when required.

EXAMPLE (2).—*Divide* $15x^4 - 8x^3 + 15$ *by* $5x^2 + 4x - 3$.

$$
\begin{array}{l}
5x^2 + 4x - 3)15x^4 - 8x^3 + 15(3x^2 - 4x + 5 \\
\qquad\qquad\quad 15x^4 + 12x^3 - 9x^2 \\
\hline
\qquad\qquad\qquad - 20x^3 + 9x^2 \\
\qquad\qquad\qquad - 20x^3 - 16x^2 + 12x \\
\hline
\qquad\qquad\qquad\qquad 25x^2 - 12x + 15 \\
\qquad\qquad\qquad\qquad 25x^2 + 20x - 15 \\
\hline
\qquad\qquad\qquad\qquad\qquad - 32x + 30
\end{array}
$$

Quotient, $3x^2 - 4x + 5$; remainder, $- 32x + 30$

Note that the 15 from the dividend is not "brought down" till we come to the last step of the process, as it is not required before.

EXAMPLE (3).—*Divide* $a^3b - a^3c + b^3c - b^3a + c^3a - c^3b$ *by* $a^2 - b^2 + ac - bc$.

Arrange the terms of both divisor and dividend as explained in the preceding paragraph.

$$
\begin{array}{l}
a^2 + ac - b^2 - bc)a^3b - a^3c - ab^3 + ac^3 + b^3c - bc^3(ab - ac - bc + c^2 \\
\qquad\qquad\qquad a^3b + a^2bc - ab^3 - ab^2c \\
\hline
\qquad\qquad\qquad - a^3c - a^2bc + ab^2c + ac^3 + b^3c - bc^3 \\
\qquad\qquad\qquad - a^3c - a^2c^2 + ab^2c + abc^2 \\
\hline
\qquad\qquad\qquad\qquad - a^2bc + a^2c^2 - abc^2 + ac^3 + b^3c - bc^3 \\
\qquad\qquad\qquad\qquad - a^2bc \qquad\quad - abc^2 \qquad + b^3c + b^2c^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad a^2c^2 \qquad + ac^3 \qquad - b^2c^2 - bc^3 \\
\qquad\qquad\qquad\qquad\qquad a^2c^2 \qquad + ac^3 \qquad - b^2c^2 - bc^3
\end{array}
$$

It is important in this example to arrange each remainder in the correct order; in the preceding examples they came naturally in the correct order.

Notice also that, where the order is troublesome, it is well to bring down the whole of the dividend at one step, instead of bringing it down term by term as required.

It is not absolutely necessary to arrange either dividend, divisor, or remainder in correct order. *It is sufficient to pick out that term of the dividend (or of the remainder) which ought to be the first*, and to divide it by the term *which ought to be the first* in the divisor, in order to get the successive terms of the quotient.

EXAMPLE (4).—*Divide* $27b^3 - 8c^3 - a^3 - 18abc$ *by* $3b - 2c - a$.

$$\begin{array}{l}
3b-2c-a)\,27b^3 - 8c^3 - a^3 - 18abc \qquad (a^2+3ab-2ac+9b^2+6bc+4c^2 \\
\qquad\qquad -a^3 \qquad + 3a^2b - 2a^2c \\
\hline
27b^3 - 8c^3 \qquad - 18abc - 3a^2b + 2a^2c \\
\qquad\qquad + 9ab^2 - 6abc - 3a^2b \\
\hline
27b^3 - 8c^3 - 9ab^2 - 12abc \qquad + 2a^2c \\
\qquad\qquad\qquad - 6abc + 4ac^2 + 2a^2c \\
\hline
27b^3 - 8c^3 - 9ab^2 - 6abc - 4ac^2 \\
27b^3 \qquad - 9ab^2 \qquad\qquad - 18b^2c \\
\hline
\qquad - 8c^3 \qquad - 6abc - 4ac^2 + 18b^2c \\
\qquad\qquad - 12bc^2 - 6abc \qquad + 18b^2c \\
\hline
\qquad - 8c^3 + 12bc^2 \qquad - 4ac^2 \\
\qquad - 8c^3 + 12bc^2 \qquad - 4ac^2
\end{array}$$

The term which *ought to be first* in the dividend is $-a^3$; the term which *ought to be first* in the divisor is $-a$; dividing $-a^3$ by $-a$ we obtain a^2; this is the first term in the quotient. Obtain the first remainder in the usual way.

The term which ought to stand first in the first remainder is $-3a^2b$; dividing this by $-a$ we obtain $+3ab$ as the second term in the quotient; and so on.

EXAMPLES.—XLVI.

Divide—

1. $x^3 + 16x^2 + 11x + 6$ by $x + 1$.

2. $x^3 - 6x^2 + 11x - 6$ by $x^2 - 3x + 2$. **3.** $x^4 - 5x^2 + 4$ by $x^2 - 3x + 2$.

4. $12x^4 - 4x^3 - 45x^2 + 113x - 56$ by $3x^2 - 7x + 8$.

5. $7x^3 - 6x^4 + 16x^2 - 59x + 42$ by $2x^2 - 5x + 6$.

6. $4x^2 + 22x - 8 - 17x^3 - 6x^4$ by $2 - 4x - 3x^2$.

7. $12x^4 - 4x^3 - 45x^2 + 100x$ by $4x^2 + 8x - 7$.

8. $6x^4 - 7x^3 - 16x^2 + 20x + 30$ by $3x^2 + 4x - 7$.

9. $12x^4 + 34x^3 - 8x^2 + 35x + 21$ by $3x^2 + 4x - 2$.

10. $x^3 + 12x^2y + 44xy^2 + 48y^3$ by $x + 2y$.

11. $8x^3 - 24x^2y + 22xy^2 - 6y^3$ by $4x^2 - 6xy + 2y^2$.

12. $81x^4 - 45x^2y^2 + 4y^4$ by $9x^2 - 9xy + 2y^2$.

13. $6x^5 - 11x^4 - 12x^3 + 22x^2 - 3x - 2$ by $2x^2 - 5x + 2$.

14. $12x^5 + 31x^4y - 40x^3y^2 + 23x^2y^3 - 38xy^4 + 12y^5$ by $4x^3 - 3x^2y + 2xy^2 - 3y^3$.

15. $a^3 - b^3 - c^3 - 3abc$ by $a - b - c$.

16. $a^3 - 8b^3 + 1 + 6ab$ by $a - 2b + 1$.
17. $a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 + 2abc$ by $a + c$.
18. $b^3c - bc^3 + c^3a - ca^3 + a^3b - ab^3$ by $b^2c - bc^2 + c^2a - ca^2 + a^2b - ab^2$.
19. $x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2$ by $y^2 + 2yz + z^2 - x^2$.
20. $b^4c - bc^4 + c^4a - ca^4 + a^4b - ab^4$ by $b^2c - bc^2 + c^2a - ca^2 + a^2b - ab^2$.
21. $36a^2b + 80b^2c + 75c^2a + 48ab^2 + 100bc^2 + 45ca^2 + 120abc$ by $9a^2 + 12ab + 15ac + 20bc$.
22. $\cdot 09a^2 - \cdot 25b^2$ by $\cdot 3a + \cdot 5b$.
23. $1\cdot 44a^2b^2 - 1\cdot 69c^2$ by $1\cdot 2ab - 1\cdot 3c$.
24. $\cdot 04a^2 + \cdot 09b^2 + \cdot 16 + \cdot 12ab - \cdot 16a - \cdot 24b$ by $\cdot 2a + \cdot 3b - \cdot 4$.
25. $\cdot 09a^2 + \cdot 16b^2 + \cdot 25c^2 - \cdot 24ab - \cdot 3ac + \cdot 4bc$ by $\cdot 4b + \cdot 5c - \cdot 3a$.

CHAPTER VIII.

ON MISCELLANEOUS OPERATIONS IN ELEMENTARY ALGEBRA.

69. On the Simplification of Expressions involving Brackets.

EXAMPLE (1).—*Simplify* $(3a + 2b) + (4b - 5c) + (2c - 2a)$.

The three quantities included within brackets are to be added together. This could be done by the method of § 56; but the method of arranging in columns is not essential to the process. Instead, we may proceed thus—

$3a$ in first bracket added to $-2a$ in the last bracket gives $+a$; $+2b$ in the first bracket added to $+4b$ in the second gives $+6b$; $-5c$ in the second bracket added to $+2c$ in the third bracket gives $-3c$. Hence we obtain the result—

$$+a + 6b - 3c$$

This is equivalent to omitting the brackets and applying the method of § 57.

EXAMPLE (2).—*Simplify* $(2a + 3b - 4c) - (3a - 2b + 4c) + (2a - 6c)$.

Here we must subtract the second bracket from the first, and add the third to the result. But the process of subtraction is to change the sign of each term, and then add (see § 59).

We may, therefore, *remove the brackets*, provided that we *change the signs* of the terms in the negative bracket; thus—

$$\begin{aligned} & (2a + 3b - 4c) - (3a - 2b + 4c) + (2a - 6c) \\ = & \ 2a + 3b - 4c - 3a + 2b - 4c + 2a - 6c \\ = & \ 2a - 3a + 2a + 3b + 2b - 4c - 4c - 6c \\ = & \ a + 5b - 14c \end{aligned}$$

Thus we have the following

RULE.—**A pair of brackets preceded by the sign + may be removed; a pair of brackets preceded by the sign − may be removed, provided the sign of each term contained within the brackets is reversed.**

EXAMPLE (3).—*Simplify* $(4x^2 - 5xy) + 3y^2 - (2xy - 3x^2 - y^2) + (3x^2 - 2y^2)$.

This gives—

$$\begin{aligned} & 4x^2 - 5xy + 3y^2 - 2xy + 3x^2 + y^2 + 3x^2 - 2y^2 \\ = & \ 4x^2 + 3x^2 + 3x^2 - 5xy - 2xy + 3y^2 + y^2 - 2y^2 \\ = & \ 10x^2 - 7xy + 2y^2 \end{aligned}$$

EXAMPLE (4).—*Simplify* $3x^3 + 5x(2x^2 - 7x + 3) - 3x(x^2 + 2x - 7)$.

Here we must multiply the first bracket by $5x$ (see § 47), and multiply the second bracket by $3x$. The first result is to be added, and the second result subtracted; the *subtraction is performed, as usual, by changing the signs.* Thus—

$$\begin{aligned} & 3x^3 + 5x(2x^2 - 7x + 3) - 3x(x^2 + 2x - 7) \\ &= 3x^3 + 10x^3 - 35x^2 + 15x - 3x^3 - 6x^2 + 21x \\ &= 3x^3 + 10x^3 - 3x^3 - 35x^2 - 6x^2 + 15x + 21x \\ &= 10x^3 - 41x^2 + 36x \end{aligned}$$

EXAMPLES.—XLVII.

Simplify—

1. $(2a + 3b - 4c) + (3a - 4b + 2c) - (4a - 2b - 3c)$.
2. $x + 2y + (3z - 4x) - (5y + 6z) + (x + 2y + 3z)$.
3. $(2p + 3q + 4r) - p - 2q + 3r - (2p - 3q - 4r)$.
4. $a + 2b + 3(c - 2a) + 4(b + 3c)$.
5. $3a(a + 2b + c) - 4b(3a - 2b + 3c) + 2c(a - 4b - 5c)$.
6. $4x^2(3x - 2y + 7) - 2x(3x^2 + 5xy + 7y^2) + 5(x^3 - 2y^3)$.
7. $2(3x^2 + 4x - 5) + 3(2x^2 - x - 7) - 4(x^2 + 2x + 7)$.
8. $p(2q - 3r + 4s) - 2q(3r - 4s + p) + 3r(4s - p + 2q) - 4s(p - 2q + 3r)$.
9. $x(x + 2y - 3z) - x^2 + 2y(2y - 3z + x) - 4y^2 + 3z(3z - x - 2y) - 9z^2$.

70. EXAMPLE (1).—*Simplify* $3(x^2 - 2x + 7) - (x + 1)(x + 3)$.

Here we must multiply $(x^2 - 2x + 7)$ by 3, and also $x + 1$ by $x + 3$, and subtract the second result from the first; *i.e.* we must change the signs in the second result, and add it to the first.

Multiplying $x + 1$ by $x + 3$ by long multiplication, we obtain $x^2 + 4x + 3$. Thus—

$$\begin{aligned} & 3(x^2 - 2x + 7) - (x + 1)(x + 3) \\ &= 3x^2 - 6x + 21 - x^2 - 4x - 3 = 2x^2 - 10x + 18 \end{aligned}$$

EXAMPLE (2).—*Simplify*—

$$(3x^2 - 2)(x + 3) - (2x^2 + 1)(x - 2) - \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 4x + 3}.$$

(i.) Multiplying $3x^2 - 2$ by $x + 3$, we obtain $3x^3 + 9x^2 - 2x - 6$.

(ii.) Multiplying $2x^2 + 1$ by $x - 2$, we obtain $2x^3 - 4x^2 + x - 2$.

(iii.) Dividing $x^3 - 6x^2 + 11x - 6$ by $x^2 - 4x + 3$ (by long division), we obtain $x - 2$.

The second and third results must be written with their signs all changed, as they are to be subtracted. Thus we have—

$$3x^3 + 9x^2 - 2x - 6 - 2x^3 + 4x^2 - x + 2 - x + 2 = x^3 + 13x^2 - 4x - 2$$

EXAMPLE (3).—*Simplify* $(x + 3)^3 - x(x + 3)^2 + x^2(x + 3) - x^3$.

(i.) To find $(x + 3)^3$, we must multiply $x + 3$ by $x + 3$, and the result again by $x + 3$. This will be found to give $x^3 + 9x^2 + 27x + 27$.

(ii.) To find $x(x + 3)^2$, we must multiply $x + 3$ by $x + 3$, and the result by x. This will be found to give $x^3 + 6x^2 + 9x$. Thus—

$$\begin{aligned} & (x + 3)^3 - x(x + 3)^2 + x^2(x + 3) - x^3 \\ &= x^3 + 9x^2 + 27x + 27 - x^3 - 6x^2 - 9x + x^3 + 3x^2 - x^3 = 6x^2 + 18x + 27 \end{aligned}$$

EXAMPLES.—XLVIII.

Simplify—

1. $(3x-2)(2x-3)-(x-6)(6x-1)$.

2. $(4x+3)(3x+4)-(2x+3)(3x+2)-6(x^2+1)$.

3. $x(2y+3z)+2y(3z+x)+3z(x+2y)-(x+2y+3z)^2$.

4. $(x+2y+5z)(x-2y+5z)-(x-2y-5z)(x+2y-5z)$.

5. $4(x+3)^2-2(x+3)(2x+3)+(2x+3)^2$.

6. $(2x-1)(2x+1)(2x+3)-8x(x-1)(x+1)$.

7. $3(3x+7)(3x+5)(x+1)-(3x+5)^3$.

8. $(a+2b+3c)+(2a+3b+c)-(3a-b-2c)+\dfrac{a^3+b^3+c^3-3bac}{a+b+c}$.

9. $\dfrac{(5x^2-7x+8)^2-(3x^2+2x-4)^2}{(5x^2-7x+8)-(3x^2+2x-4)}$.

10. $\dfrac{(6x^2-7x+10)^2-(2x^2-4x-5)^2}{(6x^2-7x+10)+(2x^2-4x-5)}$.

11. $(x+3)^4-(x-3)^4$.

12. $\dfrac{(x^2-3x+2)(x^2-5x+6)}{x^2-4x+4}$.

71. Brackets within Brackets.—In simplifying brackets within brackets, we perform the operations indicated, **starting from the inner brackets and working outwards.**

EXAMPLE (1).—*Simplify* $(x-1)\{(2x+3)^2-(2x+1)(2x+5)\}$.
By multiplying out we find that $(2x+3)^2=4x^2+12x+9$
and $(2x+1)(2x+5)=4x^2+12x+5$

Thus the given expression reduces to

$$\begin{aligned}(x-1)\{4x^2+12x+9-4x^2-12x-5\} &= (x-1)\{4\}\\ &= (x-1)\times 4\\ &= 4x-4\end{aligned}$$

EXAMPLE (2).—*Simplify* $5(3x^2+2x-5)-2\{2x^2-4x+3x[5-2(4+x)]\}$
This gives—

$$\begin{aligned}&5(3x^2+2x-5)-2\{2x^2-4x+3x[5-8-2x]\}\\ &=5(3x^2+2x-5)-2\{2x^2-4x+15x-24x-6x^2\}\\ &=15x^2+10x-25-4x^2+8x-30x+48x+12x^2\\ &=23x^2+66x-25\end{aligned}$$

EXAMPLES.—XLIX.

Simplify—

1. $\{x^2-3x(2-x)\}+(x+3)(x+5)$.

2. $\{x^2(x^2-3)-x(x^3-4)\}-2(x-2)(x-5)$.

3. $3(x-1)(2x+1)(3x-5)-\{(2x+3)(x+1)-(x+3)(x-1)\}(2x-1)$.

4. $(a+2b)(b+2c)(c+2a)-\{(a+b+c)^2-2(ab+bc+ca)\}(a+b+c)$.

5. $(a^2+b^2+c^2)(a+b+c)-\{(a+b+c)^2(a+b)+abc\}$.

6. $\{2a+3b+4c-(a+2b+4c)\}\{a(a+b+c)-2ab+b^2-ac\}$.

7. $\{\cdot 3(p^2+q^2+r^2)-\cdot 2(p+q+r)^2\}(p+q+r)$.

8. $\cdot 3\{a(\cdot 3a-\cdot 4b+\cdot 5c)+b(\cdot 2a-\cdot 3b-\cdot 4c)+c(\cdot 1a+\cdot 2b+\cdot 3c)\}$.

9. $\cdot 2x^2-\cdot 3x+\cdot 4)(\cdot 2x+\cdot 3)-\{\cdot 12+\cdot 04x(x^2+4x-5)\}$.

10. $\{a^2(2b-3c)+4b^2(3c-a)+9c^2(a-2b)\}\div\{(a-2b)(2b-3c)\}$.
11. $3x\{2-x[5+2(x-3)]\}+2x(x^2-5x-4)$.
12. $3a\{3+2[(b+2c)-a]\}-4a(2b+3c)$.
13. $2x^2\{3[2x-4(x-3)]-4[x+3-2(x-3)]\}+x\{x(x+3)-2x(4-x)\}$.
14. $(2x-3y)\{(2x+3y)(2x-3y)-[x(4x-5y)+y^2]\}$.
15. $(a+b)(b+c)(c+a)-\{a^2(b+c)+b[b(c+a)+c(a+b)]\}$.
16. $(p+q+r)(p^2+q^2+r^2)-\{p^2(p+q+r)-q^2(p+q)+r^2(q+r)\}$.

72. To find the L.C.M. of a Series of Terms.

RULE.—**The L.C.M. of a series of terms is the L.C.M. of the coefficients multiplied by the highest power of each letter that occurs in any term.**

EXAMPLE (1).—*Find the L.C.M. of* $3a^2b$, $14ab^3c$, $6a^2bcd$, $21\ abc^3d$.

L.C.M. of 3, 14, 6, and 21 is 42

the highest power of		a which occurs is		a^2
,,	,,	b	,,	b^3
,,	,,	c	,,	c^3
,,	,,	d	,,	d

Thus the L.C.M. is $42a^2b^3c^3d$.

EXPLANATION.—To find the L.C.M. of these terms is to find the lowest quantity into which each of the given terms will divide. Now, by the rules of § 64, it is obvious that each of the given terms will divide into $42a^2b^3c^3d$; and also that they would not all divide into any lower quantity. For instance, *no coefficient less than* 42 *would be divisible by each of the given coefficients;* also if we took any of the letters to a lower power, some of the given terms would not divide into the expression; thus if we took $42a^2bc^3d$, the second term would not divide into this because the second term contains b^3. Hence this expression is the L.C.M. required.

EXAMPLE (2).—*Find the L.C.M. of* $4a^3b^2c$, $12abcd$, $28bcd^3e$, $21acef^3$.

By Arithmetic we find the L.C.M. of the coefficients to be 84. Following the rule we write down the L.C.M. as—

$$84a^3b^2cd^3ef^3$$

73. Addition and Subtraction of Easy Fractions may now be performed by following the rules of Arithmetic.

EXAMPLE (1).—*Simplify* $\frac{x}{2y}+\frac{3y}{5x}-3$.

Writing the integer 3 as a fraction $\frac{3}{1}$, we obtain $\frac{x}{2y}+\frac{3y}{5x}-\frac{3}{1}$.

The L.C.M of the denominators $2y$, $5x$, 1 is $10xy$.

Dividing the first denominator into the L.C.M., we obtain $5x$, and multiplying the result by the numerator x, we obtain $5x^2$. Dividing the second denominator into the L.C.M. and multiplying by the second numerator, we obtain $6y^2$. Dividing the third denominator into the L.C.M. and multiplying by the third numerator, we obtain $30xy$.

Thus the result is $\frac{5x^2+6y^2-30xy}{10xy}$; which cannot be further simplified.

EXAMPLE (2). $\frac{x^2}{3} - \frac{x-1}{4} + \frac{x^2-2x-3}{6} + \frac{x}{2}$.

The L.C.M. of the denominators is 12. Following the arithmetical rules—

In the first fraction, $12 \div 3 \times x^2$ gives $4x^2$;
,, second ,, $12 \div 4 \times (x-1)$ gives $3(x-1)$;
,, third ,, $12 \div 6 \times (x^2-2x-3)$ gives $2(x^2-2x-3)$.
,, fourth ,, $12 \div 2 \times x$ gives $6x$.

Thus we obtain—

$$\frac{4x^2 - 3(x-1) + 2(x^2-2x-3) + 6x}{12}$$

Simplifying the numerator, we have—

$$\frac{4x^2 - 3x + 3 + 2x^2 - 4x - 6 + 6x}{12}$$

which, on further simplification, becomes—

$$\frac{6x^2 - x - 3}{12}$$

Note particularly that the second and third numerators have to be included in brackets, because in each case it is the *whole numerator* which is to be multiplied by a number, and then added to or subtracted from the other numerators; therefore to indicate these operations correctly we must enclose these numerators in brackets.

EXAMPLE (3).—*Simplify* $6x - \frac{x^2-4}{12x} + \frac{3(x-2)}{4} - \frac{5(3x+2)}{3} + \frac{x^2-7}{6x}$.

$$\frac{6x}{1} - \frac{x^2-4}{12x} + \frac{3(x-2)}{4} - \frac{5(3x+2)}{3} + \frac{x^2-7}{6x}$$
$$= \frac{72x^2 - (x^2-4) + 9x(x-2) - 20x(3x+2) + 2(x^2-7)}{12x}$$
$$= \frac{72x^2 - x^2 + 4 + 9x^2 - 18x - 60x^2 - 40x + 2x^2 - 14}{12x}$$
$$= \frac{22x^2 - 58x - 10}{12x}$$
$$= \frac{11x^2 - 29x - 5}{6x}$$

Note that in the case of the third fraction, we divide the denominator 4 into the L.C.M. of the denominators, viz, $12x$. The result $3x$ is to be multiplied by the numerator $3(x-2)$, and the product may be written $9x(x-2)$. For—

$$3x \times 3(x-2) = 3 \times x \times 3 \times (x-2)$$
$$= 3 \times 3 \times x \times (x-2)$$
$$= 9x(x-2)$$

The final step in the simplification is to reduce the fraction to its lowest terms, by dividing both numerator and denominator by 2.

EXAMPLES.—L.

Find the L.C.M. of—

1. $3x^2, 5x^3y, 2xy^3, 6xy$.
2. $14xyz, 7x^2y^2, 4y^3z^3, 2z^4x^4$.
3. $12a, 15b, 20c, 6a^2b, 8ab^2$.
4. $12, 14, 21xy, 28yz$.
5. $3xyz, 9x^3, 12y^2z, 4yz^4, 18xyz$.
6. $3x^2, 4y^2, 5z^2, 6xy, 8yz, 12zx^2$.

Simplify—

7. $\frac{x}{3} - \frac{2x}{5} + \frac{4x}{15} - \frac{7x}{30}$.
8. $\frac{x+3}{3} - \frac{2x-5}{5} + \frac{4x-15}{15} - \frac{7x+30}{30}$.

9. $3x - 2y - \frac{3(x+y)}{5} + \frac{5(x-2y)}{3} - \frac{11x-12y}{6}$.

10 $\frac{a}{b} - 2 + \frac{b}{a}$.
11. $\frac{p^2}{q} + \frac{q^2}{r} + \frac{r^2}{p} - \frac{pq}{r} - \frac{qr}{p} - \frac{rp}{q}$.

12. $\frac{3(x^2-2x-3)}{4x} - \frac{4(x+3)}{3} + \frac{x^2-2x-8}{12x} + 6$.

13. $\frac{x}{2} - \frac{y}{3} + \frac{x}{4} - \frac{y}{6} + \frac{4(x-3y)}{3} - \frac{5(2x-y)}{6}$.

14. $\frac{a}{2b} + \frac{b}{2c} + \frac{c}{2a} - \frac{a^2c + b^2a + c^2b}{2abc}$.

15. $\frac{p}{3q} + \frac{q}{3r} + \frac{r}{3p} - \frac{p^2r + q^2p + r^2q}{4pqr}$.

16. $\frac{2x(x+5)}{3} - \frac{3x(2x-3)}{8} + \frac{3(3x-5)}{12} - \frac{7}{4}$.

17. $\frac{3x^2 - 4x + 5}{3x} + \frac{5x-7}{5} - \frac{x^2 - x + 10}{10x} + \frac{5x}{3}$.

18. $\frac{2(b-c)}{b^2c^2} + \frac{2(c-a)}{c^2a^2} + \frac{2(a-b)}{a^2b^2} - \frac{a^2(b-c) + b^2(c-a) + c^2(a-b)}{a^2b^2c^2}$.

74. On Multiplication and Division of Simple Fractions.—Remembering that the separate letters which appear in the same term are all factors, we may multiply and divide simple fractions by the rules of Arithmetic.

EXAMPLE (1).—*Simplify* $\frac{3x}{y} \times \frac{12y}{5} \div \frac{x}{10}$.

Inverting the last fraction, we obtain—

$$\frac{3x}{y} \times \frac{12y}{5} \times \frac{10}{x} = \frac{3 \times x \times 12 \times y \times 10}{y \times 5 \times x}$$

But the factors x and y in the denominator cancel with the same factors in the numerator; also the factor 5 divides into the factor 10, giving the quotient 2. The product, therefore, reduces to—

$$\frac{3 \times 12 \times 2}{1} = 72$$

EXAMPLE (2).—*Simplify.* $\frac{xy}{3} \div \frac{x^5}{10y^5} \times \frac{12}{y^2} \div \frac{5x}{3y}$.

Inverting the fractions which are preceded by the sign of division—

$$\frac{xy}{3} \times \frac{10y^5}{x^5} \times \frac{12}{y^2} \times \frac{3y}{5x} = \frac{x \times y \times 10 \times y^5 \times 12 \times 3 \times y}{3 \times x^5 \times y^2 \times 5 \times x}$$

Here the factors 3 and x in the denominator cancel with the same factors in the numerator; the 5 in the denominator divides the 10 in the numerator, giving 2; and the y^2 in the denominator divides the y^5 in the numerator, giving y^3. Representing this cancelling in the usual arithmetical way, we have—

$$\frac{\cancel{x} \times y \times \overset{2}{\cancel{10}} \times \overset{y^3}{\cancel{y^5}} \times 12 \times \cancel{3} \times y}{\cancel{3} \times x^5 \times \cancel{y^2} \times \cancel{5} \times \cancel{x}} = \frac{24y^5}{x^5}$$

EXAMPLE (3).—*Simplify* $\frac{3a}{4b} \times \frac{5b^2}{6c^2} \times \frac{8c^3}{9a^3} \div \frac{4d^4}{3a^4}$.

$$\frac{3a}{4b} \times \frac{5b^2}{6c^2} \times \frac{8c^3}{9d^3} \times \frac{3a^4}{4d^4} = \frac{\cancel{3} \times a \times 5 \times \overset{b}{\cancel{b^2}} \times \overset{4}{\cancel{8}} \times \overset{c}{\cancel{c^3}} \times \cancel{3} \times a^4}{4 \times \cancel{b} \times \underset{2}{\cancel{6}} \times \cancel{c^2} \times \underset{3}{\cancel{9}} \times d^3 \times \cancel{4} \times d^4} = \frac{5a^5bc}{12d^7}$$

EXAMPLES.—LI.

Simplify—

1. $\frac{3a^2}{5b^2} \times \frac{3b^2}{5c^2} \times \frac{100c^4}{81a^4}$.

2. $\frac{4xy}{5z} \times \frac{3yz}{4x} \div \frac{9y^3}{10}$

3. $\frac{a}{b} \div \frac{2c}{3d} \times \frac{4e}{5f} \div \frac{8ace}{15bdf}$.

4. $\frac{[illegible]}{3c} \times \frac{[illegible]}{5a} \times \frac{[illegible]}{7b}$.

5. $\frac{a}{3bc} \times \frac{b}{5ca} \times \frac{c}{7ab} \div \frac{1}{9a^2b^2c^2}$.

6. $\frac{8pq^2}{9r^3s^4} \times \frac{2qr^2}{3s^3p^4} \div \frac{5p^3q^4}{6rs^2} \div \frac{4q^3r^4}{9sp^2}$.

7. $\frac{3l^3m^5}{4n^6} \times \frac{8m^2n^3}{9l^5} \div \frac{20n^4l^2}{3m^4}$.

8. $\frac{7x^3}{y^2z^4} \times \frac{8y^5}{x^2z^2} \div \frac{14xy}{z^6}$.

[In the following examples the rule of § 51 must be strictly obeyed :—]

9. $\frac{3a}{5b} \times \frac{2c}{9d} + \frac{3b}{5a} \times \frac{2d}{9c} - \frac{4d}{9c} \div \frac{8a}{27b}$.

10. $\frac{2}{5b} \div \frac{3}{2a} - \frac{6}{5c} \div \frac{3}{10a} + \frac{4}{bc} \times \frac{3a}{8}$.

11. $\frac{3a}{2b} \times \frac{3b}{4c} \div \frac{3a}{2c} - \frac{5a}{3c} \times \frac{2b}{7a^2} + \frac{3a}{4b} \div \frac{6c}{b}$.

12. $\frac{20pq}{9r^2} \times \frac{6qr}{5p^2} - \frac{3pq}{5r^2} \times \frac{2qr}{9p^2} \div \frac{3pr}{10q^2} - \frac{5p}{qr}$.

13. $\frac{p}{q}\left(\frac{r}{p} - \frac{q}{s}\right) - \frac{3q}{r}\left(\frac{2p}{q} - \frac{r}{3s}\right)$.

14. $\frac{3a}{bc}\left(\frac{2ab}{c} - \frac{5ac}{b}\right) - \frac{5a^2}{b^2} \div \frac{3c^2}{a^2}$.

15. $\frac{2}{3}\left(\frac{p}{q} + \frac{r}{s}\right) - \frac{3}{5}\left(\frac{p}{r} - \frac{q}{s}\right) + \frac{5p}{2r}\left(\frac{q}{s} - \frac{r}{4}\right)$.

75. On Squaring.—We can find the square of any quantity by multiplying it by itself.

EXAMPLE (1).—*Find the square of* $-4ab^2c^3$.

Following the rules of § 60, $-4ab^2c^3$ multiplied by $-4ab^2c^3$ gives $+16a^2b^4c^6$.

If the quantity to be squared contains more than one term, this involves a long multiplication. But in this case there is a fairly simple rule, which enables us to write down the answer.

RULE.—To square a quantity which contains more than one term, write down the square of each term, followed by twice the product of every pair of terms. The square of each term is positive; the product of each pair of terms follows the ordinary rule of signs.

EXAMPLE (2). $(2a-3b+4c)^2 = 4a^2+9b^2+16c^2-12ab+16ac-24bc.$

Here the first three terms are the squares of $2a$, $-3b$, and $4c$ respectively; the fourth term is twice the product of $2a$ and $-3b$; the fifth term is twice the product of $2a$ and $+4c$, and the sixth term is twice the product of $-3b$ and $+4c$.

EXAMPLE (3). $(5a-3b+c-d)^2 = 25a^2+9b^2+c^2+d^2-30ab+10ac-10ad-6bc+6bd-2cd.$

Here the first four terms are the squares of the given terms; the fifth term $= 2\times(5a)\times(-3b)$; the sixth term $= 2\times(5a)\times c$; the seventh term $= 2\times(5a)\times(-d)$; and so on.

EXPLANATION.—The reason for this rule will be obvious if the student multiplies $5a-3b+c-d$ by itself by the process of long multiplication. The *square of each term will occur, and will occur once only;* thus the square of $-3b$ will occur as the second term in the second line of multiplication.

(Note also that the square of $-3b$ is $+9b^2$.)

Again, the *product of any pair of terms will occur, and occur twice*; thus the product of $-3b$ and $-d$ appears both as the fourth term in the second line of multiplication, and as the second term in the fourth line of multiplication.

EXAMPLES.—LII.

Write down the square of—

1. $3a-2b$. **2.** $c-5d$. **3.** $2a-7c$. **4.** $2a-3b+4c$.
5. $3x-2y+5z$. **6.** $3x^2-2x+3$. **7.** $5x^2-7x-3$.
8. $2a-3b-4c+5d$. **9.** $3p-2q+5r-4s$.
10 x^3-3x^2+3x-5. **11.** $2x^3-x^2+2x-5$.
12. $\cdot 2x+\cdot 3y$. **13.** $\cdot 5x-\cdot 4y$.
14. $\cdot 2x-\cdot 3y+\cdot 4z$. **15.** $2p-\cdot 5q+\cdot 25r$.

76. On Square Root.—The process of extracting square root in Algebra is practically the same as in Arithmetic. It is essential here, as in long division, to *take account of the correct order of the terms.*

EXAMPLE (1).—*Find the square root of* $9x^4-48x^3+106x^2-112x+49$.

$$
\begin{array}{lll}
 & 9x^4-48x^3+106x^2-112x+49 & (3x^2-8x+7 \\
 & 9x^4 & \\
\hline
6x^2-8x) & -48x^3+106x^2 & \\
 & -48x^3+64x^2 & \\
\hline
6x^2-16x+7) & 42x^2-112x+49 & \\
 & 42x^2-112x+49 &
\end{array}
$$

Here the given expression is in the correct order. The square root of $9x^4$ is $3x^2$; we place $3x^2$ in the root, and subtract its square from the given expression (the "**radicand**"), and bring down as many terms as are likely to be required for the next step.

Doubling $3x^2$, we obtain the trial divisor, $6x^2$; dividing $6x^2$ into $-48x^3$, we obtain the next term in the root, viz. $-8x$, which is also placed after the trial divisor to form the true divisor. We multiply the true divisor by $-8x$, and subtract. We obtain the next trial divisor by doubling the last term in the last true divisor, thus obtaining $6x^2 - 16x$; and so on, following the ordinary arithmetical rule.

EXAMPLE (2).—*Extract the square root of* $a^4+4b^4+9c^4-12b^2c^2-6a^2c^2+4a^2b^2$.

$$\begin{array}{rl} & a^4 + 4b^4 + 9c^4 - 12b^2c^2 - 6a^2c^2 + 4a^2b^2(a^2 + 2b^2 - 3c^2 \\ & a^4 \\ \hline 2a^2 + 2b^2) & 4b^4 + 9c^4 - 12b^2c^2 - 6a^2c^2 + 4a^2b^2 \\ & +4b^4 \qquad\qquad\qquad\qquad\qquad + 4a^2b^2 \\ \hline 2a^2 + 4b^2 - 3c^2) & +9c^4 - 12b^2c^2 - 6a^2c^2 \\ & +9c^4 - 12b^2c^2 - 6a^2c^2 \end{array}$$

The order in the radicand is not correct, though a^4 is correctly in the first place. In the first remainder, the term which *ought to stand first* is $4a^2b^2$; we therefore divide the trial divisor $2a^2$ into this term, and thus obtain $2b^2$ as the next term in the root. In the second remainder, the term which *ought to stand first* is $-6a^2c^2$; dividing this by $2a^2$, we obtain the third term in the root, viz. $-3c^2$.

EXAMPLES.—LIII.

Extract the square root of—

1. $9a^2 - 6ab + b^2$.
2. $16a^2 + 24abc + 9b^2c^2$.
3. $4x^2 - 20xy + 25y^2$.
4. $81a^2b^2 - 90abc + 25c^2$.
5. $25p^2 + 70pqr + 49q^2r^2$.
6. $9x^4 - 30x^3 + 85x^2 - 100x + 100$.
7. $25x^4 + 30x^3y - 61x^2y^2 - 42xy^3 + 49y^4$.
8. $49x^4 + 28x^3 - 52x^2 - 16x + 16$.
9. $9a^2 + 4b^2 + c^2 - 12ab + 6ac - 4bc$.
10. $9p^2 + 25q^2 + 49r^2 + 30pq - 42pr - 70qr$.
11. $25x^2 + 16y^2 + 9z^2 - 24yz + 30zx - 40xy$.
12. $\cdot09x^2 + \cdot04y^2 + \cdot16z^2 - \cdot12xy + \cdot24xz - \cdot16yz$.
13. $4x^2 + \cdot09y^2 + \cdot01z^2 + 1\cdot2xy - \cdot4xz - \cdot06yz$.

Find the fourth root of—

14. $81x^4 - 216x^3 + 216x^2 - 96x + 16$.
15. $256x^4 + 1280x^3 + 2400x^2 + 2000x + 625$.

77. The letters of an algebraical expression may denote negative quantities.—In this case great care must be taken to observe the rules of signs in evaluating the expression.

EXAMPLE (1).—*Evaluate* x^3 *if* $x = -3$.
$x^3 = x \times x \times x = (-3) \times (-3) \times (-3) = (+9) \times (-3) = -27$.

It should be noticed that an even power of any negative

quantity will always be positive; but that any odd power of a negative quantity will always be negative.

This result depends merely on the rule of signs in multiplication, and is obvious from the consideration of the following example :—

EXAMPLE (2).—*Find all the powers of* -5 *as far as the sixth power.*

$$
\begin{array}{rl}
-5 & \\
-5 & \\
\hline
+25 & = \text{the square of } -5 \\
-5 & \\
\hline
-125 & = \text{the cube of } -5 \\
-5 & \\
\hline
+625 & = \text{the fourth power of } -5 \\
-5 & \\
\hline
-3125 & = \text{the fifth power of } -5 \\
-5 & \\
\hline
+15625 & = \text{the sixth power of } -5
\end{array}
$$

In each step we multiply by a negative quantity, and therefore *reverse the sign* of the preceding result.

EXAMPLE (3).—*If* $a=2$, $b=-3$, $c=4$, $d=-5$, *evaluate* $a^2-3b^2+4cd-4abc$.

Here—

$$a^2 = 4$$
$$b^2 = (-3) \times (-3) = +9$$
$$\therefore -3b^2 = -27$$
$$4cd = 4 \times 4 \times (-5) = -80$$
$$-4abc = -4 \times 2 \times (-3) \times 4 = +96$$

Thus we have $4 - 27 - 80 + 96$, which is equal to -7.

EXAMPLES.—LIV.

If $a = 2$, $b = -3$, $c = -4$, $d = 1$, $e = 0$, evaluate—

1. $a^2 + b^2 + d^2 + 2ab + 2ad + 2bd$.

2. $(a - b)^2 - c^2 - d^2 + 2cd - 3ce + 4de$.

3. $a^3 + b^3 + c^3 - 3abc$.

4. $(c + d + e)(-c + d + e)(c - d + e)(c + d - e)$.

5. $(a + b + c)^2 - \{2(b + c) - (3d + 4e)\}^2$.

6. $(abc + bcd + cde + dea + eab)^2 - \dfrac{ab + bc + cd}{bc + cd + de}$.

7. $\dfrac{a + b + c}{a + d + e} - \dfrac{b + c + d}{c + d + e} + \dfrac{3ab}{cd}$.

8. $(a^2 + b^2 + c^2)/(a + b + c)$.

If $a = 1$, $b = -{\cdot}2$, $c = {\cdot}3$, $d = -1$, evaluate—

9. $ab + bc + cd + da$.

10. $3ab + 4bc - 5cd - 6da$.

11. $(a + b)^2 - (b + c + d)^2$.

12. $(a + b + c)(-a + b + c)(a - b + c)(a + b - c)$.

13. $a^2 + b^2 + c^2 - \dfrac{a^3 + b^3 + c^3}{a + c + d}$.

14. $3a\{b + 2c(d - 3a)^2\}$.

15. $(2a + 5b + d)^2 + (3a - 5b + d)(a^2 + b^2 + c^2)$.

CHAPTER IX.

ON THE SOLUTION OF EQUATIONS.

78. On Solving an Equation.

EXAMPLE (1).—*Find the value of x, given that* $3x + 11 = 26$.

We are given that, when 11 is added to $3x$, the result is 26.
It follows that the value of $3x$ must be 15.
But if 3 times x is 15, the value of x must be 5.

The above is a very easy example of "**solving an equation.**" The statement $3x + 11 = 26$ is the **equation to be solved;** to "solve" it, means to *find that value of x for which the equation is true.* Any easy equation can always be solved by Arithmetic and common sense, and a little practice in this is of very great value, before learning the mechanical rules employed in Algebra.

EXAMPLE (2).—*Solve the equation* $\frac{2x}{5} + 7 = 12$.

When 7 is added to $\frac{2x}{5}$, the result is 12; therefore the value of $\frac{2x}{5}$ must be 5.

If 2-fifths of x is 5
1-fifth of x must be 2·5
and therefore x must be $5 \times 2{\cdot}5$; *i.e.* 12·5

EXAMPLE (3).—*Solve the equation* $\frac{2x}{5} - 3 = x - 9$.

The result of subtracting 3 from $\frac{2x}{5}$ is the same as the result of subtracting 9 from x.

It follows that x must be greater than $\frac{2x}{5}$ by 6.

But the difference between x and $\frac{2x}{5}$ is $\frac{3x}{5}$. Thus the value of $\frac{3x}{5}$ is 6.

If 3-fifths of x is 6
1-fifth of x must 2
and therefore x must be 10

EXAMPLES.—LV.

Solve the following equations by the methods of this paragraph :—

1. $3x = 21$. **2.** $5x = 125$. **3.** $2x = 3$.
4. $3x = \cdot 2$. **5.** $2x + 5 = 15$. **6.** $3x + 4 = 25$.
7. $5x - 4 = 21$. **8.** $2x + 5 = 10$. **9.** $3x + 7 = 8$.
10. $\frac{x}{5} = 3$. **11.** $\frac{x}{7} = 4$. **12.** $\frac{2x}{7} = 16$.
13. $\frac{2x}{5} = 3$. **14.** $\frac{3x}{5} = \frac{2}{3}$. **15.** $\frac{x}{5} + 7 = 9$.
16. $\frac{x}{3} + 5 = 12$. **17.** $\frac{2x}{3} - 7 = 11$. **18.** $\frac{3x}{5} + 20 = 44$.
19. $\frac{4x}{7} - 3 = 13$. **20.** $3x - 5 = x - 1$.
21. $5x + 5 = x + 29$. **22.** $4x - 2 = 2x + 6$.
23. $\frac{2x}{3} + 5 = 2x - 15$. **24.** $\frac{3x}{5} + 10 = x + 4$.

79. Equations which do not contain Fractions.—In more complicated equations the methods of the preceding paragraph become too tedious. The following rule reduces the work to an easy mechanical process. Note that the two expressions which are separated by the sign =, are called the two "sides" of the equation.

RULE.—(a) **Work out any brackets there may be.**

(b) **Place *all* terms which contain *x* on the left-hand side of the equation; and all terms without *x* on the right-hand side. In this operation, terms which are moved from one side of the equation to the other must have their signs changed.**

(c) **Simplify each side.**

(d) **Divide each side by the coefficient of *x*.**

EXAMPLE (1).—*Solve* $3(2x + 3) - 5(x + 7) = 4(x - 20) + 24$.

(*a*) $6x + 9 - 5x - 35 = 4x - 80 + 24$

(*b*) $6x - 5x - 4x = -9 + 35 - 80 + 24$

[Note that we change the sign of $4x$ when we carry it from the right to the left side of the equation; and we change the signs of $+9$ and -35 when we carry them from the left to the right side of the equation.]

(*c*) $-3x = -30$

(*d*) Dividing each side by -3, we obtain—

$$x = 10$$

EXPLANATION.—Taking the four steps in order—

(*a*) requires no explanation.

(*b*) We have found that—

$$6x + 9 - 5x - 35 = 4x - 80 + 24$$

If now we omit $4x$ from the right-hand side, and insert $-4x$ on the left-hand side, *we have diminished both sides by* $4x$, and thus we have not destroyed the equality. We now have—

$$6x + 9 - 5x - 35 - 4x = -80 + 24$$

If we now omit -35 from the left-hand side, and insert $+35$ on the right-hand side, *we have increased both sides by* 35; and thus the results are still equal. Thus—

$$6x + 9 - 5x - 4x = +35 - 80 + 24$$

Similarly, we may omit $+9$ from the left-hand side, and insert -9 on the right-hand side, as this is equivalent to diminishing both sides by 9.

(*c*) requires no explanation.

(*d*) is obviously correct, and is technically justified by the rule that "if equals are divided by equals, the quotients are equal."

EXAMPLE (2). $3(x+1)(x+2) - 2(x+2)(x+3) = (x+3)(x+4)$.

Working out the brackets, we obtain—

$$3x^2 + 9x + 6 - 2x^2 - 10x - 12 = x^2 + 7x + 12$$

Bringing terms which contain x to the left-hand side, and the others to the right-hand side—

$$3x^2 + 9x - 2x^2 - 10x - x^2 - 7x = -6 + 12 + 12$$

Simplifying: $-8x = +18$

Dividing each side by -8: $x = -\frac{9}{4}$

the sign of the result being determined by the law of signs for division.

EXAMPLE (3). $\cdot 3(2x+4) + \cdot 7(x-1) = 2\cdot 5x$.

Clearing of brackets: $\cdot 6x + 1\cdot 2 + \cdot 7x - \cdot 7 = 2\cdot 5x$

Transposing terms: $\cdot 6x + \cdot 7x - 2\cdot 5x = -1\cdot 2 + \cdot 7$

Simplifying: $-1\cdot 2x = -\cdot 5$

Dividing both sides by $-1\cdot 2$: $x = +\frac{\cdot 5}{1\cdot 2}$

i.e. $x = \cdot 41\dot{6}$

EXAMPLE (4). $3(a+1)(a+3) = 3(4a-2) + 20$.

Clearing of brackets: $3a^2 + 12a + 9 = 12a - 6 + 20$

Transposing the terms: $3a^2 + 12a - 12a = -9 - 6 + 20$

Simplifying: $3a^2 = 5$

Dividing by 3:— $a^2 = \frac{5}{3}$

i.e. $a^2 = 1\cdot\dot{6}$

But if a is the quantity which is equal to $1\cdot\dot{6}$ when squared, a must be the square root of $1\cdot\dot{6}$ (see § 33). Extracting this square root by the method of § 36, we obtain—

$a = 1\cdot 291$ almost exactly

It should be noted at this point that, strictly speaking, the **sign of a square root is always ambiguous.** Thus the square root of 16 may be **either** $+4$ or -4; for if -4 be multiplied by -4 the result is $+16$. Thus theoretically $\sqrt{16} = \pm 4$; but for practical purposes we take $\sqrt{16} = +4$.

EXAMPLES.—LVI.

Solve the equations—

1. $3(x-7) = x+5$.

2. $3(x+2) = 2(x-3) + 17$.

3. $2x + 9 = 7x - 6$.

4. $5x + 7 = 8x - 8$.

5. $\cdot 3x + \cdot 2 = \cdot 5x - \cdot 2$.

6. $x - 2 = \cdot 2x + \cdot 6$.

7. $3x + 7 = \cdot 2x + 35$.

8. $\cdot 3(x+3) = \cdot 5(x+1)$.

9. $\cdot 2(2x+3) = \cdot 5(x-2) + 1$.

10. $3\{x - 3(2x + 3)\} = 4 - 5(3x + 2)$.
11. $4\{5x - 4(x - 1)\} = 3\{4(x + 1) - 7x\}$.
12. $(2x + 1)(2x + 3) = (x + 1)(x + 3) + 3x^2 + 20$.
13. $3(x + 1)(x + 4) + 5 = 5x(x + 3) + 9$.
14. $(2x + 3)(3x - 2) + 12 = (x + 6)(6x + 1)$.
15. $(5x + 2)(2x - 5) = (5x - 2)(2x + 8) - 5(3 + 10x)$.
16. $(3x + 1)^2 = (3x - 5)(3x + 7) + 2x - 7$.
17. $3(2x + 3)^2 = 12(x + 3)^2 - (x + 11)$.
18. $(2x + 1)(2x + 5) - (2x + 3)^2 = (4x + 1)(4x + 5) - (4x + 3)^2$.
19. $(2x + 7)(3x - 5) + 12 = (6x + 7)(x - 5) + 10x$.
20. $x(x + 2)(x + 4) = x(x^2 + 8) - (3x^2 - 81)$.
21. $(x - 1)(x - 3)(x - 5) = x(x^2 - 5) + x(33 - 9x)$.
22. $(p + 1)(p + 2) + (p + 2)(p + 3) = 2(p + 1)(p + 3) + 2p^2$.
23. $(a - 1)(a - 3) + (a - 3)(a - 5) = 2(a - 1)(a - 5) + 2a^2$.
24. $\cdot 135(a - \cdot 12) = \cdot 105(a + \cdot 12)$.
25. $\cdot 38(2a + 5) = \cdot 16(a + 20)$.
26. $\cdot 012(x + 5) = \cdot 01(2x - 5)$.
27. $2\cdot 3(3\cdot 5x - 1\cdot 5) = 1\cdot 5(2\cdot 5x + 1\cdot 5)$.
28. $3q^2 + 25 - 2(q^2 + 10) = 4(q^2 + 8) - 5q^2$.

80. Equations containing Fractions.—If an equation contains fractions we multiply **both sides** by the L.C.M. of **all the denominators**; this will get rid of all the fractions, and we may then proceed as in the last paragraph.

EXAMPLE (1). $\frac{x + 3}{4} - \frac{x}{2} = \frac{x + 4}{6} - 2.$

The L.C.M. of the denominators 4, 2, 6 is 12. Multiplying each term by 12—

$$\frac{x + 3}{\cancel{4}} \times \frac{\overset{3}{\cancel{12}}}{1} = 3(x + 3)$$

$$\frac{x}{\cancel{2}} \times \frac{\overset{6}{\cancel{12}}}{1} = 6x$$

$$\frac{x + 4}{\cancel{6}} \times \frac{\overset{2}{\cancel{12}}}{1} = 2(x + 4)$$

$$2 \times 12 = 24$$

The equation now stands—

$$3(x + 3) - 6x = 2(x + 4) - 24$$

And this is easily solved by the methods of the last paragraph, the result being—

$$x = 5$$

It is easily seen from the above example that in multiplying each fraction by the L.C.M. of the denominators we may use the rule, "**Divide the denominator into the L.C.M.; multiply the result by the numerator.**"

EXAMPLE (2). $\frac{x^2 + 10}{3x} - \frac{6x - 5}{12} = \frac{1}{3} + \frac{x - 10}{12x}.$

The L.C.M. of the denominators is $12x$. Following the rule just given, we obtain—

$$4(x^2 + 10) - x(6x - 5) = 4x + (x - 10)$$

Clearing of brackets and transposing, we obtain—

$$4x^2 - 6x^2 + 5x - 4x - x = -40 - 10$$
$$i.e. \quad -2x^2 = -50$$
$$\therefore x^2 = 25$$
$$\therefore x = \pm 5$$

EXAMPLE (3).—

$$\frac{3a}{8}\left\{1 - \frac{2}{3}\left(\frac{2}{a} + \frac{1}{a^2}\right)\right\} + \frac{1}{8a^2}(3 - 2a) = a\left\{\frac{1}{3} - \frac{1}{2a}\left(1 - \frac{3}{a^2}\right)\right\}.$$

Clearing of brackets—

$$\frac{3a}{8}\left\{1 - \frac{4}{3a} + \frac{2}{3a^2}\right\} + \frac{3}{8a^2} - \frac{1}{4a} = a\left\{\frac{1}{3} - \frac{1}{2a} + \frac{3}{2a^3}\right\}$$

$$i.e. \quad \frac{3a}{8} - \frac{1}{2} + \frac{1}{4a} + \frac{3}{8a^2} - \frac{1}{4a} = \frac{a}{3} - \frac{1}{2} + \frac{3}{2a^2}$$

Multiplying by the L.C.M. of the denominators, which is $24a^2$—

$$9a^3 - 12a^2 + 6a + 9 - 6a = 8a^3 - 12a^2 + 36$$

Transposing—

$$9a^3 - 12a^2 + 6a - 6a - 8a^3 + 12a^2 = -9 + 36$$
$$i.e. \quad a^3 = 27$$
$$\therefore a = \sqrt[3]{27}$$
$$= 3$$

EXAMPLES.—LVII.

1. $\frac{x}{3} + \frac{3x + 2}{8} = \frac{5x - 1}{4} - \frac{7}{12}$.
2. $\frac{3x + 2}{6} - \frac{2x - 3}{4} = \frac{5 - 4x}{3} + \frac{3}{4}$.
3. $\frac{3a}{5} - \frac{3 - 2a}{2} = \frac{3a + 5}{10} + \frac{3(a - 1)}{5}$.
4. $\frac{3a - 8}{14} - \frac{5a}{3} = 2 - \frac{4a + 1}{21}$.
5. $\frac{3(2x - 3)}{5} + \frac{8x - 7}{25} = 3 + \frac{3x - 2}{10}$.
6. $\frac{2x - 7}{9} + \frac{5(2 - x)}{12} = \frac{5}{6} - \frac{3x - 8}{4}$.
7. $\frac{4p - 3}{2} + \frac{5(p - 7)}{7} = \frac{3}{4} - \frac{2p - 3}{2}$.
8. $\frac{3c - 2}{4} + \frac{6c - 5}{2} = \frac{c + 8}{3} + 10$.
9. $\frac{4x - 1}{3} + \frac{3}{5} = \frac{2x + 5}{7} - \frac{4}{15}$.
10. $\frac{1}{2}(x + 3) - \frac{2}{3}(x - 2) = \frac{1}{4}(3x + 1) - 2$.
11. $\frac{1}{5}\left(\frac{x}{4} - \frac{2}{3}\right) + \frac{2}{5}\left(\frac{x}{3} - \frac{1}{3}\right) = \frac{x - 3}{2} - \frac{2}{3}$.
12. $\frac{p^2 - 5}{2} + \frac{3 - p^2}{3} = \frac{2p^2 - 3}{5} - \frac{3p^2 - 6}{7}$.

13. $\frac{2x^2-5}{9}+\frac{x^2-7}{3}=\frac{3x^2-8}{5}-2.$

14. $\frac{x^2+1}{2}-\frac{x^2+3}{4}=\frac{x^2}{5}-\frac{x^2-5}{6}.$

15. $\frac{x(x+1)}{4}-\frac{x(x+1)}{6}=\frac{x+1}{2}-\frac{5x+2}{12}.$

16. $\frac{x}{3}\left(\frac{x}{2}-\frac{5}{4}\right)-\frac{3x}{4}(x-\frac{1}{2})=\frac{x+1}{3}-\frac{3(x+4)}{8}.$

17. $\frac{x}{3}\left\{\frac{1}{2}-\left(\frac{x}{2}+\frac{2}{3}\right)\right\}-\frac{x}{6}\{2-\frac{3}{2}(x-4)\}=\frac{x^2-1}{12}.$

18. $\frac{a}{4}\left\{2-3\left(\frac{4}{a}-\frac{5}{a^2}\right)\right\}+\frac{a}{3}\left\{3-\frac{1}{3}\left(\frac{3}{a}-\frac{2}{a^2}\right)\right\}=\frac{3a}{2}+\frac{4}{a}-\frac{1}{6}\left(\frac{1}{6a}+\frac{20}{a^2}\right).$

81. On Equations which contain Two Unknown Letters.—An equation which contains two unknown letters will not determine the value of either letter: but **it enables us to find a relation between the two letters which is often useful.** The rule is the same as for ordinary equations, except that, after the clearing of any fractions or brackets which the equation may contain, we place all terms containing the one letter on the left-hand side, and all terms not containing this one letter on the right-hand side.

EXAMPLE (1).—*Solve* $2+5x-4y=4x-3y+6$.
On the left-hand side place only those terms which contain x.

$$+5x-4x=-2+4y-3y+6$$

Simplifying: $$x=y+4$$

This is the solution of the equation, and means that the original equation will be true, provided that the numerical value assigned to x is greater by 4 than that assigned to y.

For instance, if we make $x=7$ and $y=3$, then—

$$2+5x-4y=2+35-12=25$$

and $$4x-3y+6=28-9+6=25$$

Hence the original equation is true, or (to use the more common expression) the original equation is "**satisfied.**"

EXAMPLE (2).—*Solve the equation* $3+4(a-3b)=5+6(5b-3a)$.
Clearing of brackets: $3+4a-12b=5+30b-18a$.
Transposing: $4a+18a=-3+12b+5+30b$
Simplifying: $22a=42b+2$

Both sides of this equation can be divided by 2; this does not destroy the equality, and reduces the equation to lower terms—

$$11a=21b+1$$

Thus— $$a=(21b+1)\div 11$$

$$=\frac{21b+1}{11}$$

We have here "**expressed *a* in terms of *b*;**" *i.e.* we have obtained an expression for a which involves b. If we wished to express b in terms of a, we must, in transposing, place only terms which contain b on the left-hand side. We then obtain—

$$-12b - 30b = -3 - 4a + 5 - 18a$$
$$i.e. \quad -42b = -22a + 2$$

Dividing through by -2 (in order to make the coefficient of b positive, and at the same time to reduce the equation to lower terms)—

$$21b = 11a - 1$$
$$\therefore \; b = \frac{11a - 1}{21}$$

In many cases these equations will give the ratio of the one letter to the other.

EXAMPLE (3).—*Find the ratio of* $p : q$, *given that* $p + 2q - 4(p - q) = (3p - 2q)$.

Then—
$$p + 2q - 4p + 4q = 3p - 2q.$$
$$\therefore \; p - 4p - 3p = -2q - 4q - 2q$$
$$i.e. \quad -6p = -8q$$
$$\therefore \; p = (-8q) \div (-6) = \tfrac{4}{3}q$$

But if p is $\frac{4}{3}$ of q, the ratio of $p : q$ is $4 : 3$ (see § 18).

EXAMPLE (4).—*Find the ratio of x to y if* $\cdot 2\frac{x}{y} - \cdot 3\frac{y}{x} = \cdot 35\frac{y}{x} - \frac{x}{y}$.

Notice that $\cdot 2\frac{x}{y} = \frac{\cdot 2}{1} \times \frac{x}{y} = \frac{\cdot 2x}{y}$; and similarly for the other terms. Thus the equation may be written—

$$\frac{\cdot 2x}{y} - \frac{\cdot 3y}{x} = \frac{\cdot 35y}{x} - \frac{x}{y}.$$

To clear of fractions, we multiply by the L.C.M. of the denominators, which is xy. Thus—

$$\cdot 2x^2 - \cdot 3y^2 = \cdot 35y^2 - x^2$$

Transposing:
$$\cdot 2x^2 + x^2 = \cdot 3y^2 + \cdot 35y^2$$
$$i.e. \; 1\cdot 2x^2 = \cdot 65y^2$$
$$\therefore \; x^2 = \frac{\cdot 65}{1\cdot 2}y^2$$

Using decimal division, we have—
$$x^2 = \cdot 541\dot{6}y^2$$

Taking the square root of both sides—
$$x = y\sqrt{\cdot 541\dot{6}}$$
$$\text{and since } \sqrt{\cdot 541\dot{6}} = \cdot 73598 \text{ very approximately}$$
$$\therefore \; x = \cdot 73598y$$

i.e. the ratio of $x : y$ is $\cdot 73598 : 1$.

It is well to note at this point that in dealing with equations *we may perform any operation on one side of an equation, provided we perform the same operation on the other*, so that the equality of the two sides is maintained.

For instance, in this last example we have extracted the square root of both sides. Similarly, we may divide both sides by the same quantity; we may extract the cube root of both sides; we may change all the signs on both sides; and so on.

EXAMPLE (5).—*Find the ratio of $x:y$, given that* $2(5y^3-45x^3)=3(15x^3-10y^3)$.

Clearing of fractions: $10y^3 - 90x^3 = 45x^3 - 30y^3$

Divide both sides by 5 (to reduce the figures as far as possible)—

$$2y^3 - 18x^3 = 9x^3 - 6y^3$$

Transpose: $-18x^3 - 9x^3 = -2y^3 - 6y^3$

Change the signs on both sides (to save the trouble of working with minus signs): $18x^3 + 9x^3 = 2y^3 + 6y^3$

$$i.e.\ 27x^3 = 8y^3$$

Take cube root of both sides: $3x = 2y$

$$\therefore x = \tfrac{2}{3}y$$

$$\therefore x:y = 2:3$$

EXAMPLES.—LVIII.

Find x in terms of y, given that—

1. $3x - 5y = 2y + 2x - 8$.
2. $2(x+3) + 3y = 3(3y+x) + 4$.
3. $3\{x + 2(y-4)\} = 2\{y - 7(x-1)\} + 10(x - y - 1)$.
4. $\frac{2}{3}\left(\frac{x}{4y} - \frac{3}{5}\right) = \frac{3}{5}\left(\frac{10x}{9y} - \frac{15}{2y}\right)$.

Find q in terms of p, given that—

5. $\frac{p+3}{5} - \frac{q+5}{10} = \frac{p-q}{4} + 3$.
6. $\frac{1}{3}(p - 2q) + \frac{1}{4}(3p - 4q) = \frac{1}{2}(2p - 3q)$.
7. $\cdot45q = \cdot78p - \cdot234$. (Divide each term by ·45, by decimal division.)
8. $\cdot23(p + 2q) = \cdot56(3q - 2p + 4)$.
9. $\cdot25(1\cdot12p - \cdot36q + 4) = \cdot12(1\cdot25p + \cdot25q + 5)$.

Find the ratio of $a:b$ from the following equations:—

10. $3a - 5b = 20b - 12a$.
11. $5b = 10a - 9b$.
12. $\cdot23a - \cdot48b = \cdot35b + \cdot1a$.
13. $5\{7a - 4(b - 2a)\} = a - 3(2b - a)$.
14. $\cdot123(\cdot4a + b) = \cdot234(\cdot6a + \cdot1b)$.
15. $5\{\cdot23a + \cdot45(b - a)\} = \cdot7\{\cdot34b - \cdot12(a - b)\}$.
16. $1\cdot345(a^2 - b^2) = \cdot681(a^2 + b^2)$.
17. $\frac{3}{7}(a - b)(a + b) = \frac{1}{20}(a + b)^2 - \frac{ab}{10}$.
18. $\frac{3a}{b}\left\{2 - \frac{b}{a}\left(\frac{a}{b} - 2\right)\right\} = \frac{3a^2}{b^2} - \frac{2a}{b} + \frac{5a}{b}\left(\frac{a}{b} + 1\right)$.
19. $\frac{2a}{3b}\left\{\frac{5b}{a} - 3\left(2 + \frac{3b}{a}\right)\right\} = \frac{a^2}{2b^2} - \frac{4a}{b}\left(1 + \frac{a}{b}\right)$.
20. $a = 3\cdot235\frac{a^2 - b^2}{a}$.
21. $b^2 = \frac{a^3 + \cdot936b^3}{b}$.
22. $a = 1\cdot368\frac{a^4 - b^4}{a^3}$.

82. On the Solution of Problems by means of Equations.—A very large number of problems can be solved by setting them out as algebraical equations.

EXAMPLE (1).—*Find a number the sum of whose sixth and seventh parts is equal to* 26.

Let x represent the number required. Then its sixth part is $\frac{x}{6}$; and its seventh part is $\frac{x}{7}$.

Thus x is to have such a value that $\frac{x}{6} + \frac{x}{7} = 26$.

The value of x is therefore found by solving this equation (see § 78).

Clearing of fractions: $7x + 6x = 1092$

$$i.e.\ 13x = 1092$$
$$\therefore x = 84$$

Note carefully the method adopted. The question stated that—

$\frac{1}{6}$ *of required number* + $\frac{1}{7}$ *of required number* = 26

This statement is "*translated*" (so to speak) into the algebraical equation—

$$\frac{x}{6} + \frac{x}{7} = 26$$

which equation is easily solved.

EXAMPLE (2).—*Find two numbers which differ by* 20, *and whose ratio is* 5 : 9.

Let x represent the smaller number, then $x + 20$ represents the larger (because the larger is to be 20 more than the smaller). Since their ratio is 5 : 9, the smaller is $\frac{5}{9}$ of the greater.

$$i.e.\quad x = \tfrac{5}{9}(x + 20)$$
$$\text{hence } x = \frac{5x}{9} + \tfrac{100}{9}$$
$$\therefore 9x = 5x + 100$$
$$\therefore 4x = 100$$
$$\therefore x = 25$$

But x represents the smaller number, and $x + 20$ represents the larger number.

Thus the two numbers required are 25 and 45.

Summary of Method.—Numbers which differ by 20 can be represented by x and $(x + 20)$.

The question states that x is to $(x + 20)$ in the ratio of 5 : 9.

$$\text{Hence } x = \tfrac{5}{9}(x + 20)$$

Solve this equation.

EXAMPLE (3).—*Divide* 100 *into two parts, such that five times one part exceeds three times the other part by* 4.

We can represent the two parts by x and $(100 - x)$; for the second part would be found by subtracting the first part from 100.

The question states that

5 times x exceeds 3 times $(100 - x)$ by 4

Translating this into an equation, we have—

$$5x - 3(100 - x) = 4$$

Solving:

$$5x - 300 + 3x = 4$$
$$\therefore 8x = 304$$
$$\therefore x = 38$$

Thus the one part is 38; and therefore the other part is 62.

EXAMPLE (4).—*From a certain number we subtract its tenth part; to the result we add* 2; *the last result is multiplied by* $\frac{4}{5}$, *and then subtracted from the original number. The remainder is* 60. *Find the original number.*

Represent the original number by x. Subtracting its tenth part, we obtain $x - \frac{x}{10}$. Adding 2, we obtain $x - \frac{x}{10} + 2$. Multiplying by $\frac{4}{5}$, and subtracting the product from x, we obtain $x - \frac{4}{5}\left(x - \frac{x}{10} + 2\right)$. Since the result is known to be 60, it follows that—

$$x - \frac{4}{5}\left(x - \frac{x}{10} + 2\right) = 60$$

Solving this equation:

$$x - \frac{4x}{5} + \frac{2x}{25} - \frac{8}{5} = 60$$
$$\therefore 25x - 20x + 2x - 40 = 1500$$
$$\therefore 7x = 1540$$
$$\therefore x = 220$$

EXAMPLES.—LIX.

1. Find a number whose third part exceeds its fourth part by 5.

2. Find a number the sum of whose third and fourth parts is 21.

3. Find a number such that three-fifths of the number exceeds two-sevenths of the number by 33.

4. Find two numbers which differ by 40, and whose ratio is 3 : 8.

5. Find two numbers which differ by 12, and whose ratio is 11 : 14.

6. Find two numbers whose sum is 105, and whose ratio is 7 : 8.

7. Find two numbers whose sum is 65, and whose ratio is 6 : 7.

8. Find two numbers whose sum is 50, and whose difference is 12.

9. Find two numbers whose sum is 60, and whose difference is 56.

10. Find two numbers in the ratio 5 : 8, such that three times the second exceeds four times the first by 12. (Represent the two numbers by x and $\frac{8x}{5}$.)

11. Divide 50 into two parts, such that twice the one part exceeds three times the other part by 75.

12. Divide 120 into two parts, such that if both parts be increased by 15, they are in the ratio 3 : 2.

13. Divide 200 into two parts, such that if the one part be diminished by 60, and the other by 35, the results are in the ratio 2 : 1.

14. Two numbers are in the ratio 2 : 3; and when each is increased by 15, they are in the ratio 5 : 6. Find them.

15. Two numbers are in the ratio 3 : 5; if the first is doubled, and then diminished by 10, the result is equal to the second. Find the numbers.

16. Two quantities are in the ratio 4 : 7; if each is subtracted from 3, the results are in the ratio 10 : 1. Find the quantities.

17. A certain number is subtracted from 100; the remainder is multiplied by 2; the product is subtracted from the original number; the remainder is 40. Find the original number.

18. A certain number is subtracted from 120; the remainder is trebled, and then increased by 20; it is again trebled, and then diminished by the original number; the result is 140. Find the original number.

19. Two numbers differ by 13; the sum of three times the smaller and five times the larger is 105. Find them.

20. Two quantities differ by ·123; the sum of three times the smaller and five times the larger is 1. Find them.

21. Find three quantities, such that the sum of the first and second is 2·5; the sum of the second and third is 3·25; and the sum of the first, second, and twice the third is 5.

22. The ratio of two quantities is 2 : 3, and the difference of their squares is 180. Find them.

23. The ratio of two quantities is 3 : 4; the sum of their squares is 225. Find the difference of their squares.

24. Two numbers are in the ratio 3 : 5; when 30 is added to each, they are in the ratio 4 : 5. Find the numbers.

83. Harder Problems.

EXAMPLE (1).—*Two vessels of water each contain* 10 *gallons. How much water must be transferred from one vessel to the other, in order that the quantities of water in the two vessels should be in the ratio* 5 : 7?

Let x gallons represent the quantity that must be transferred.

After taking x gallons from the first vessel, it contains $(10 - x)$ gallons.

After putting x gallons into the second vessel, it contains $(10 + x)$ gallons.

Thus it is required that $(10 - x)$ and $(10 + x)$ should be in the ratio 5 : 7.

$$\text{Hence} \qquad (10 - x) = \tfrac{5}{7}(10 + x)$$
$$\therefore 7(10 - x) = 5(10 + x)$$
$$\therefore -12x = -20$$
$$\therefore x = \tfrac{5}{3}$$

Answer, $\frac{5}{3}$ gallon.

EXAMPLE (2).—*A debt of* £3 14*s.* 6*d. is paid entirely in half-crowns and florins. If the total number of coins paid is* 33, *how many are there of each coin?*

Let x represent the number of half-crowns; then $(33 - x)$ represents the number of florins.

Thus x half-crowns and $(33 - x)$ florins amount to £3 14*s.* 6*d.*

$$\text{But} \qquad x \text{ half-crowns} = 5x \text{ sixpences}$$
$$(33 - x) \text{ florins} = 4(33 - x) \text{ sixpences}$$
$$£3\ 14s.\ 6d. = 149 \text{ sixpences}$$
$$\text{Thus} \qquad 5x + 4(33 - x) = 149$$
$$\text{Whence} \qquad x = 17$$

Note that *we must reduce each sum of money to the same denomination*, before we can form the algebraical equation. For otherwise the numbers will not form the equation.

For instance, 2 half-crowns + 3 florins = 11 shillings; but the number 2 + the number 3 is not equal to the number 11.

EXAMPLE (3).—*Find, in yards, the length and breadth of a rectangular field whose area is* 1 *acre; given that the ratio of the length to the breadth is* 5 : 2.

Let x yards be the length; then the breadth is $\frac{2}{5}$ of the length, *i.e.* $\frac{2x}{5}$ yards. The area is the product of the length and breadth; *i.e.* $x \times \frac{2x}{5}$; *i.e.* $\frac{2x^2}{5}$ sq. yards. But the area is 1 acre, *i.e.* 4840 sq. yards.

$$\therefore \frac{2x^2}{5} = 4840$$
$$\therefore 2x^2 = 24,200$$
$$\therefore x^2 = 12,100$$
$$\therefore x = 110$$

Thus the length is 110 yards, and the breadth 44 yards.

EXAMPLE (4).—*The distance between London and Cambridge is* 52 *miles. A cyclist, who rides at* 12 *miles an hour, starts from London to Cambridge at* 9 *a.m.*; *another cyclist, who rides at* 9 *miles an hour, starts from Cambridge to London at* 10 *a.m.*: *find where they meet.*

Suppose they meet x miles from London. Then the first cyclist has ridden x miles, and the second cyclist has ridden $(52 - x)$ miles. Again, the time occupied in riding is equal to the distance travelled divided by the rate of travelling.

Thus to ride x miles at 12 miles an hour, the first cyclist takes $\frac{x}{12}$ hours; and to ride $(52 - x)$ miles at 9 miles an hour, the second cyclist takes $\frac{52 - x}{9}$ hours.

But the difference between these two times must be 1 *hour*, because the first cyclist started an hour earlier.

$$\text{Thus} \qquad \frac{x}{12} - \frac{52 - x}{9} = 1$$
$$\text{whence} \qquad 9x - 12(52 - x) = 108$$
$$\therefore 21x = 732$$
$$\therefore x = 34\tfrac{6}{7}$$

Answer, $34\frac{6}{7}$ miles from London.

EXAMPLES.—LX.

1. If one vessel contains 20 gallons of water, and another contains 12 gallons, how much must be taken from the second and put into the first, in order that the ratio of the quantities of water in the two vessels should be 7 : 1?

2. Two vessels contain equal quantities of water. If 8 gallons be taken from one vessel and put into the other, the quantities of water which they now contain are in the ratio 3 : 7. How much was there originally in each vessel?

3. Two tanks contain quantities of water which are in the ratio 5 : 7. If 50 gallons be drawn from each, the quantities left are in the ratio 9 : 13. How much did each originally contain?

4. One purse contained £100, another £60. Two equal debts were paid, one from each purse; and the sums of money left in the purses are in the ratio 7 : 3. Find the amount of the debts.

5. One purse contained £100, another £60. One debt was paid from the first purse; and another debt, of half the amount of the first debt, was paid from the second purse. Find the sums of money left in the purses, given that they are in the ratio 3 : 2.

6. A legacy was left to two brothers, the elder receiving double as much as the younger. The elder gave a quarter of his share, and the younger a fifth of his share, to a sister, who received in all £700. Find the elder brother's legacy.

7. The sum of £1000 was divided between two brothers; each put £200 in the bank. They then found that a quarter of what the first had left, together with a tenth of what the second had left, amounted to £120. How much did each receive?

8. A debt of 3*s.* 10*d.* is paid with 21 coins, some of which are sixpences and the rest pennies. How many are there of each?

9. A debt of £1 is paid with 48 coins, of which some are shillings, some sixpences, and the rest pennies. If the number of pennies is double the number of sixpences, find how many there are of each coin.

10. A debt of £9 is paid partly in crowns, partly in half-crowns, partly in florins. If the number of crowns, the number of half-crowns, and the number of florins are proportional to 5, 2, and 3, find the number of each.

11. The area of a rectangular field is 3 acres; and the ratio of the length to the breadth is 6 : 5. Find its length.

12. The breadth of a rectangular lawn is three quarters of its length; in the middle of the lawn is a flower-bed whose length and breadth are one quarter of the length and breadth of the lawn. The area of the remainder of the lawn is 4500 sq. feet. Find the length of the lawn.

13. A square lawn is entirely surrounded by a gravel path 3 feet wide. If the area of the gravel path be 276 sq. feet, find the area of the lawn.

14. The length of a tank is double its breadth, and treble its depth; the tank can just hold 36 cub. feet of water. Find its dimensions.

15. The ratio of the area of a circle to the area of the square on its radius is 3·1416; find, in inches, the radius of a circle whose area is 2 sq. feet.

16. A man's salary increases each year in the ratio 5 : 6. In three years he receives altogether £364. Find his salary in the first year.

17. Two men start running in opposite directions round a racing-track of five laps to the mile. The one runs $8\frac{1}{2}$ yards per second, and the other $7\frac{1}{2}$. Where will they meet?

18. Two men ride along the same road; the one rides at 12 miles an hour, and the other at $10\frac{1}{2}$ miles an hour. If the slower rider has half an hour's start, how far does he ride before he is overtaken?

19. Two men ride to meet one another from a distance of 100 miles apart. Their paces are 9 and 12 miles an hour respectively, and the faster rider starts an hour and twenty minutes earlier than the other. When do they meet?

20. A man in riding 100 miles starts at a certain pace, but after 25 miles increases his pace in the ratio 2 : 3; after another 45 miles he diminishes his pace in the ratio 5 : 4. The whole journey takes him 8 hours. Find the pace at which he started.

21. Two men started with equal sums of money. One invested his money at 3 per cent. simple interest, and the other invested his money at 5 per cent. simple interest. At the end of 20 years they had £1440 between them, reckoning both principal and interest. How much had they between them originally?

22. Find two quantities whose product is 40, and whose ratio is $5:3$.

84.—On Simultaneous Equations.—If we are asked to find two positive quantities whose *product is* 12, we can give an indefinite number of correct answers. For $3 \times 4 = 12$, $2 \times 6 = 12$, $\cdot 5 \times 24 = 12$, $5 \times 2\frac{2}{5} = 12$, and so on.

But if we are asked to find two positive quantities whose *product is* 12, and whose *difference is* 4, there is only one correct answer. The numbers must be 2 and 6.

In general, it will be found that we can find **two** unknown quantities, if we are given **two** statements about them, and not otherwise.

Now, each statement can be represented algebraically as an equation; hence **we can find two unknown quantities if we are given two equations which they satisfy.**

When two equations are used for this purpose they are called **simultaneous equations.**

To solve two simultaneous equations, we derive from them a new equation *which contains only one of the unknown letters*, and which can therefore be solved by the methods of the preceding paragraphs.

EXAMPLE (1).—*If* $\left\{\begin{array}{r} 3x + 2y = 12 \\ \textit{and } 5x - 3y = 1 \end{array}\right\}$ *find x and y.*

Multiplying *both sides* of the first equation by 3, and *both sides* of the second equation by 2, we obtain the two equations—

$$\begin{aligned} 9x + 6y &= 36 \\ 10x - 6y &= 2 \\ \text{Adding these two equations: } 19x \qquad &= 38 \end{aligned}$$

This last equation, **in which y does not appear,** is sufficient to determine the value of x, and gives $x = \mathbf{2}$.

Now, return to one of the original equations—say the first. Since $x = 2$, this equation gives $6 + 2y = 12$, from which we can find the value of y; for $2y = 12 - 6$, *i.e.* $2y = 6$; whence $y = \mathbf{3}$.

The above working is set out most conveniently as follows:

$$\begin{array}{rll} & 3x + 2y = 12 & \text{(i.)} \\ & 5x - 3y = 1 & \text{(ii.)} \\ \text{(i.)} \times 3: & 9x + 6y = 36 & \text{(iii.)} \\ \text{(ii.)} \times 2: & 10x - 6y = 2 & \text{(iv.)} \\ \text{(iii.)} + \text{(iv.)}: & 19x \qquad = 38 & \text{(v.)} \\ \text{from (v.)}: & \mathbf{x = 2} & \\ \therefore \text{ from (i.)}: & 6 + 2y = 12 & \text{(vi.)} \\ \text{from (vi.)}: & 2y = 6 & \\ & \therefore \mathbf{y = 3} & \end{array}$$

Answer, $x = 2$; $y = 3$.

EXPLANATION.—The student must notice that the object of multiplying equations (i.) and (ii.) by the numbers 3 and 2 was to make the coefficients of y in equations (iii.) and (iv.) equal to one another; thus equation (v.) does not contain y. This process is called "**eliminating y.**"

In this case we multiplied (i.) by the coefficient of y in (ii.), and we multiplied (ii.) by the coefficient of y in (i.). This is the simplest rule, but will sometimes make the figures unnecessarily large.

EXAMPLE (2).—*Solve* $\left.\begin{array}{l}33x + 8y = 24 \\ 22x + 7y = 21\end{array}\right\}$ . . . . (i.) . . . . (ii.)

Eliminate x: (i.) × 2: $66x + 16y = 48$ . . . . (iii.)

(ii.) × 3: $66x + 21y = 63$ . . . . (iv.)

Subtract (iii.) from (iv.): $5y = 15$ . . . . (v.)

$\therefore \quad y = 3.$

from (ii.): $22x + 21 = 21$

$\therefore \ 22x = 0$

$\therefore \ x = 0$

Note that we may eliminate whichever of the letters we please. In this example we had to subtract equation (iii.) from (iv.) in order to cancel the term in x.

To obtain the correct multipliers for equations (i.) and (ii.) in order to eliminate x, we may proceed thus—

H.C.F. of 33 and 22 = 11

$\left.\begin{array}{l}33 \div 11 = 3 \\ 22 \div 11 = 2\end{array}\right\}$ $\left\{\begin{array}{l}\text{multiply (ii.) by 3} \\ \text{multiply (i.) by 2}\end{array}\right.$

EXAMPLES.—LXI.

Solve the simultaneous equations—

1. $5x + 2y = 12.$ $8x - 3y = 13.$
2. $4x + 3y = 10.$ $3x + 5y = 13.$
3. $7x + 6y = 27.$ $11x - 3y = 30.$
4. $5x + 4y = 17.$ $7x + 2y = 13.$
5. $21x + 5y = 73.$ $14x - 11y = 20.$
6. $4x - 4y = 4.$ $5x - 6y = 2.$
7. $35p - 6q = 29.$ $21p + 2q = 23.$
8. $7c + 8d = 14.$ $21c - 20d = 42.$

85. Simultaneous equations are not always given in the same form as in the preceding paragraph; but wherever possible they should be reduced to this form before attempting to solve them.

EXAMPLE (1).— $\left.\begin{array}{c}\dfrac{x}{4} - \dfrac{y}{3} = 0 \\ \dfrac{x}{5} + \dfrac{y+1}{4} = \dfrac{x+1}{3} + \dfrac{y+5}{10}\end{array}\right\}$ . . . . (i.) . . . . (ii.)

In equation (i.), clearing of fractions—

$$3x - 4y = 0 \quad \ldots \ldots \quad \text{(iii.)}$$

In equation (ii.), clearing of fractions—

$$12x + 15y + 15 = 20x + 20 + 6y + 30$$

Collecting the terms in x and y on the left-hand side—

$$-8x + 9y = 35 \quad \ldots \ldots \quad \text{(iv.)}$$

Equations (iii.) *and* (iv.) *are now in the correct "form," and can be solved by the methods of the preceding paragraph.*

Eliminate x: (iii.) × 8: $24x - 32y = 0$ . . . . (v.)

(iv.) × 3: $-24x + 27y = 105$ . . . . (vi.)

(v.) + (vi.): $-5y = 105$

whence $y = -21$

$\therefore$ from (iii.): $3x + 84 = 0$

whence $x = -28$

EXAMPLE (2).— $\left.\begin{aligned} 3(\cdot 1x + \cdot 004) &= 4(\cdot 01 - \cdot 03y) \\ \cdot 35x &= 1\cdot 8y + \cdot 003 \end{aligned}\right\}$ . . . . . (i.) (ii.)

From (i.) : $\cdot 3x + \cdot 012 = \cdot 04 - \cdot 12y$

$\therefore \ \mathbf{\cdot 3x + 1\cdot 2y = \cdot 028}$ . . . . (iii.)

From (ii.) : $\mathbf{35x - 1\cdot 8y = \cdot 003}$ . . . . (iv.)

Eliminate x : H.C.F. of ·3 and ·35 = ·05.

$\left.\begin{aligned} \cdot 3 \div \cdot 05 &= 6 \\ \cdot 35 \div \cdot 05 &= 7 \end{aligned}\right\}$ $\left\{\begin{aligned} &\text{multiply (iv.) by 6} \\ &\text{multiply (iii.) by 7} \end{aligned}\right.$

(iii.) × 7 : $2\cdot 1x + \cdot 84y = \cdot 196$ . . . . (v.)

(iv.) × 6 : $2\cdot 1x - 10\cdot 8y = \cdot 018$ . . . . (vi.)

(v.) − (vi.) : $11\cdot 64y = \cdot 178$

whence $y = \cdot 178 \div 11\cdot 64 =$ **·0153 nearly**

Thus, from (iii.) : $\cdot 3x + \cdot 001836 = \cdot 028$

whence $x = \cdot 026164 \div \cdot 3 =$ **·0872 nearly**

EXAMPLE (3). $\dfrac{2x + y}{8} = \dfrac{x + 10}{5} = \dfrac{y - 4}{2}.$

To find x and y we must form two equations of the usual type. We are given that—

$$\frac{2x + y}{8} = \frac{y - 4}{2}$$

whence $2x + y = 4y - 16$

$\therefore \ \mathbf{2x - 3y = -16}$ . . . . (i.)

Also we are given that— $\dfrac{x + 10}{5} = \dfrac{y - 4}{2}$

whence $2x + 20 = 5y - 20$

$\therefore \ \mathbf{2x - 5y = -40}$ . . . . (ii.)

Solve (i.) and (ii.) ; eliminate x—

(i) − (ii.) : $2y = 24$

$\therefore \ \mathbf{y = 12}$

$\therefore$ from (i) : $2x - 36 = -16$

whence $\mathbf{x = 10}$

EXAMPLES.—LXII.

1. $5x = 2y.$
$3(x - 3) = y - 8.$

2. $2(x + 4) = 3(y - 5).$
$3(x + 1) = y.$

3. $\dfrac{3x}{2} = \dfrac{y + 3}{4}.$
$\dfrac{7x - 4}{5} = \dfrac{y + 3}{6}.$

4. $\dfrac{x}{2} + \frac{2}{3}(y + 1) = x + y - 2$
$7x = 2y + 4.$

5. $\dfrac{3x + 2}{10} = \dfrac{y + 2}{4}.$
$\frac{4}{7}(x + 1) = \dfrac{2y}{3}.$

6. $\cdot 3x \times \cdot 5y = \cdot 08.$
$\cdot 07x = \cdot 05y + \cdot 002.$

7. $\cdot 4x + 5y = \cdot 014.$
$\cdot 3x - y = \cdot 001.$

8. $3(x + y) = \cdot 08 - 2y.$
$\cdot 12x = \cdot 004 + \cdot 3y.$

9. $\cdot 45 = 3\cdot 4a + b.$
$a + b = \frac{3}{7}(5a + 2b).$

10.* $8xy + 7x = 23.$
$12xy - 3x = 21.$

11. $2p^2 - 3q = 3.$
$p^2 = 2q - 1.$

12.* $3x^2 - 2xy = 4.$
$5x^2 + 3xy = 32.$

13. $3a + 5 = 2c + 7 = 5a - 3c.$

14. $\frac{3x + 2y}{5} = \frac{7x - 3y}{4} = 3.$

15. $\frac{4lm - 3m}{3} = \frac{lm + 5m + 4}{2} = lm + 2.$

16. $\cdot 03x + \cdot 5y - \cdot 4 = \cdot 04x - \cdot 2y - \cdot 3 = 0.$

86. Problems leading to Simultaneous Equations.

EXAMPLE (1).—*If three men and five boys together earn £6 in a week, and if five men and two boys together earn £8 2s. in a week, find the wages of a man and a boy.*

Suppose that each man earns x shillings per week, and each boy y shillings per week. Then three men and five boys earn $(3x + 5y)$ shillings per week.

$$\therefore 3x + 5y = 120 \quad \text{(i.)}$$

for £6 = 120 shillings.

In the same way we find from the second statement that—

$$5x + 2y = 162 \quad \text{(ii.)}$$

Solving equations (i.) and (ii.), we obtain—

$$x = 30 ; y = 6$$

$\therefore$ each man earns 30 shillings, and each boy 6 shillings per week.

Note that each of the two statements of the problem gives us an equation; and the two equations are sufficient to determine the two unknowns.

EXAMPLE (2).—*By travelling at 20 miles an hour, a train covers a certain distance in a certain time. If it had travelled at 30 miles an hour, it would have covered 10 miles more in half an hour less. Find the actual distance travelled.*

Suppose that the train actually travelled x miles, and that its time for the journey was y hours. Then the statements in the problem may be set out as follows:—

(i.) Travelling at 20 miles an hour for y hours, the train went x miles.

(ii.) Travelling at 30 miles an hour for $(y - \frac{1}{2})$ hours, the train went $(x + 10)$ miles.

But we know, as in Arithmetic, that—

$$\text{distance travelled} = \text{rate} \times \text{time}$$

$$\therefore \text{from (i.)}: x = 20y \quad \text{(iii.)}$$

$$\text{and from (ii.)}\ x + 10 = 30(y - \tfrac{1}{2}) \quad \text{(iv.)}$$

Solving these equations, we obtain—

$$x = 50 ; y = 2\tfrac{1}{2}$$

Thus the distance travelled was 50 miles.

EXAMPLES.—LXIII.

1. If I can buy 3 yards of silk and 4 yards of linen for 4*s*., and 8 yards of silk and 10 yards of linen for 10*s*. 6*d*., find the price of each per yard.

2. Six cows and eleven sheep cost £93; while four cows and seven sheep cost £61: find the price of a single sheep.

---

* Eliminate xy.

3. If 4*s*. 8*d*. will purchase either 5 lbs. of apples and 12 lbs. of cherries, or 8 lbs. of apples and 8 lbs. of cherries, find the price of each kind of fruit.

4. I bought eight cows and five sheep at one market and sold them at another, and gained £13; if I had bought and sold four cows and fifteen sheep, I should have gained £14: how much did I gain on each cow and on each sheep? (Let the gain on one cow be £x.)

5. I bought eight cows and five sheep at one market and sold them at another, and gained £7; if I had bought and sold twelve cows and fifteen sheep, I should have gained £6: find the gain or loss on each cow and each sheep. (Assume that there is a *gain* in each case; a negative answer will represent a *loss*.)

6. I bought four cows and ten sheep at one market and sold them at another, and gained £2; had I bought and sold ten cows and fifty sheep, I should have lost £5: find the gain or loss on each cow and each sheep.

7. If 3 cubic centimetres of gold and 10 cubic centimetres of iron together weigh 132·02 grammes, while 7 cubic centimetres of gold and 6 cubic centimetres of iron together weigh 179·26 grammes, find the weight of 1 cubic centimetre of gold.

8. I ride a certain distance at 10 miles an hour; a friend, who starts an hour later, but rides at 12 miles an hour, is 2 miles behind me when I finish the journey: find the length of the journey.

9. Each of two trains travels a distance of 105 miles; the first train travels 5 miles an hour faster than the second, and finishes the journey in half an hour less time: find the rates of travelling.

10. A certain sum of money is divided equally between the members of a class. Had there been ten fewer in the class, each would have received 5*s*. more; had there been ten more in the class, each would have received 3*s*. less. How much did each receive?

11. Find two quantities such that the first exceeds half the second by ·6, while one quarter of the second exceeds one-fifth of the first by ·42.

12. The salaries of two officials are in the ratio 8 : 11. If each received a rise of £40 per annum, they would be in the ratio 3 : 4. Find their salaries.

13. Find two quantities such that if each is raised 50 per cent. they will differ by 60; but if the greater is lowered 20 per cent. it will exceed the smaller by 16.

14. Find the ages of two brothers, if 10 years ago their ages were in the ratio 5 : 3, and 5 years hence they will be in the ratio 4 : 3.

87. On Arithmetical Rules expressed as Algebraic Formulæ.—Any rule in Arithmetic can be expressed by an algebraic formula; the construction of such formulæ forms a very useful exercise.

EXAMPLE (1).—*Construct a formula for the sum to which a given principal will amount, if put out to simple interest for a given time at a given rate.*

Let the given principal be £**P**, the given rate **R** per cent., and the given time **T** years; and let the required amount be £**A**.

According to the arithmetical rule, we must multiply the principal by the rate and by the time, and divide the result by 100; this gives the interest. To find the result, we must add the interest to the principal. Following this rule—

We multiply **P**, **R**, and **T**: result, PRT

We divide this by 100: result, $\frac{PRT}{100}$

Add this quantity to £**P**: result $P+\frac{PRT}{100}$

The last quantity is the required amount. Hence we have the formula—

$$A = P + \frac{PRT}{100}$$

Such a formula is very useful, as it expresses concisely the relation between the quantities represented by the various letters. It can be used to find the value of any one of these quantities when the others are known; the process employed is the method of solving an equation.

EXAMPLE (2).—*Use the formula in the preceding example to answer the following question: What principal will amount to £672 in 4 years at 5 per cent. simple interest?*

In this problem—

A = the amount = £672
P = the principal, which is unknown
R = the rate per cent. = 5
T = the time = 4 years

Substituting these values of **A**, **R**, and **T** in the given formula, we obtain the equation—

$$672 = P + \frac{20P}{100}$$

Solving this equation—

$$672 = P + \frac{P}{5}$$
$$\therefore\ 3360 = 5P + P$$
$$i.e.\ \ 6P = 3360$$
$$\text{whence } P = 560$$

Thus the required principal is £560.

EXAMPLE (3).—*Two men are running a race of 1 mile. A is allowed to start a given number of yards ahead of B, and a given number of seconds before B. If it is known how many yards A runs in each second, construct a formula to determine how many yards B must run in a second in order that the result should be a dead heat.*

Suppose that A starts Y yards ahead of B, and S seconds before B, and that his pace is p yards per second. Let x yards per second be the required pace for A.

(i.) To find the distance which A has to run, we must subtract his start from 1 mile, *i.e.* from 1760 yards. Result, $1760 - Y$.

(ii.) To find the time in which A will reach the tape, we must divide the distance he has to run (in yards) by his pace (in yards per second). This gives the time in seconds. Result, $\frac{1760 - Y}{p}$.

(iii.) To find the time in which B must run the mile, we must subtract from this the number of seconds between their times of starting.

Result, $\frac{1760 - Y}{p} - S$.

(iv.) To find the pace at which B must run, we must divide 1760 yards by the time which he may take (in seconds).

Result, $1760 \div \left\{\frac{1760 - Y}{p} - S\right\}$.

Thus we obtain the formula—

$$x = 1760 \div \left\{\frac{1760 - Y}{p} - S\right\}$$

EXAMPLE (4).—*Use the preceding formula to answer the following question: A and B run a mile race; A has 200 yards' start, and runs 6 yards per second. If B runs $6\frac{2}{3}$ yards per second, find how many seconds A must be allowed to start before B, in order that the race may result in a dead heat.*

In this problem, x = B's pace = $6\frac{2}{3}$ yards per second; Y = A's start = 200 yards; S = the required number of seconds; p = A's pace = 6 yards per second.

Substituting these values of x, Y, and p in the above formula, we obtain the equation—

$$6\tfrac{2}{3} = 1760 \div \left\{\frac{1760 - 200}{6} - S\right\}$$

$$i.e. \quad \tfrac{20}{3} = 1760 \div \{260 - S\}$$

$$\therefore \quad \tfrac{20}{3} \times \{260 - S\} = 1760$$

$$i.e. \quad \tfrac{5200}{3} - \frac{20S}{3} = 1760$$

$$\text{whence } S = -4$$

Thus A must start -4 seconds before B; this *negative* result obviously means that **A must start 4 seconds after B.**

EXAMPLES.—LXIV.

Express the following three arithmetical rules as algebraic formulæ:—

1. To find the Area of the whole surface of a brick: multiply Length by Breadth, Length by Thickness, and Breadth by Thickness; add these products, and double the result.

2. To find the Diameter of a circle of given Area: divide the Area by 3·1416; take the square root of the quotient, and double the result.

3. To find the Volume of wood used in constructing a closed box: multiply the Length, Breadth, and Depth of the box; diminish the Length, and the Breadth and the Depth by twice the Thickness of the wood, and multiply the results together; subtract the second result from the first.

4. Construct a formula to find the Sale price of an article, given its Cost price and the per-centage Profit.

5. Construct a formula to find the Perimeter of a square (*i.e.* the sum of the length of its sides), given its Area.

6. A train starts from a given station and travels north at the rate of x miles an hour; y hours later, a second train starts from a station z miles south of the first station, and travels north at w miles an hour. Construct a formula to find the distance d between the two trains when the first train has been travelling p hours.

7. A and B run a race; A has h yards' start and runs k yards in each second; B runs l yards in each second. Construct a formula to find the distance (n yards) which B will run before he overtakes A.

8. Use the formula of Question 4 to determine the cost price of an article which is sold for £30 at a gain of 20 per cent.

9. Use the formula of Question 4 to determine the per-centage loss, if an article is bought for £70 and sold for £66 10*s*.

10. Use the formula of Question 7 to determine at what rate a man runs who is caught after he has run 200 yards by another man who started 16 yards behind him, and who has run at the rate of 9 yards per second.

11. Use the formula of Question 7 to find how far a man must walk at 4 miles an hour to overtake another walking at $3\frac{1}{2}$ miles an hour, who starts 7 furlongs ahead of him.

12. A man's income is £I per annum; the portion he spends is to the portion he saves in the ratio $p : q$. Construct a formula for his total savings (T) after n years.

CHAPTER X.

ON EASY FACTORS AND FRACTIONS.

88. On Monomial Factors.—It is often very important in Algebra to be able to determine the factors of a given expression, *i.e.* to find quantities whose product shall be equal to the given expression. We will consider a few of the simplest methods of factorizing. It is obvious that factorizing is the reverse process to multiplication.

The simplest case of factorizing is where one factor consists of a single term, such as $5x^2$ or $3abc^2$, etc. Such a factor is called a **monomial factor.**

EXAMPLE (1).—*Factorize* $15a^2 + 10ab - 20ac$.

In this case we see that the figure 5, and also the letter a, will divide every term. No other figure or letter will divide every term. We therefore divide by $5a$.

$$5a)\underline{15a^2 + 10ab - 20ac}$$
$$3a + 2b - 4c$$

Thus $15a^2 + 10ab - 20ac = 5a(3a + 2b - 4c)$.

Note that this is the reversed process of Ex. (1), § 61.

EXAMPLE (2).—*Factorize* $42x^5yz^2 - 56x^4y^3z^2 + 70x^3yz^5w$.

In this case each term will divide by each of the following quantities: 14, x^3, y, z^2.

$$14x^3yz^2)\underline{42x^5yz^2 - 56x^4y^3z^2 + 70x^3yz^5w}$$
$$3x^2 - 4xy^2 + 5z^3w$$

Thus $42x^5yz^2 - 56x^4y^3z^2 + 70x^3yz^5w = 14x^3yz^2(3x^2 - 4xy^2 + 5z^3w)$.

In these cases the division process is usually performed mentally.

EXAMPLES.—LXV.

Factorize the following expressions :—

1. $21a^3 - 28a^2b + 35ab^2$.
2. $9a^2b - 12ab^2 + 15b^3$.
3. $15x^5 - 25x^4y - 100x^3y^2$.
4. $12x^2yz - 20xy^2z - 80y^3z$.
5. $38x^3yz^2 - 57x^5z^4$.
6. $52p^5q^5 + 91p^2q^3r^5$.
7. $6x^6 + 30x^5 + 24x^4 + 42x^3$.
8. $18x^5 - 27x^4y + 45x^3y^2 - 36x^2y^3$.
9. $6abcd + 8bcde - 10cdef$.
10. $51abc - 34abd + 85cde$.

89. RULE.—**If we multiply the sum of any two terms by their**

difference, the product will be the difference of the squares of these terms.

ILLUSTRATIONS.—

$$\begin{array}{l} x + a \\ x - a \\ \hline x^2 + ax \\ \quad\; - ax - a^2 \\ \hline x^2 \qquad\;\; - a^2 \end{array} \qquad\qquad \begin{array}{l} 3p + 5qr \\ 3p - 5qr \\ \hline 9p^2 + 15pqr \\ \qquad\; - 15pqr - 25q^2r^2 \\ \hline 9p^2 \qquad\qquad\;\; - 25q^2r^2 \end{array}$$

Thus in the second case, we take the two terms $3p$ and $5qr$, and we multiply their sum by their difference. In the addition, two of the terms cancel, and the final product is the difference of the squares of $3p$ and $5qr$. We may therefore assume the law to be general, and write down the product in all such cases.

EXAMPLES.—$(ab + cd)(ab - cd) = a^2b^2 - c^2d^2$.
$(3x + 5y)(3x - 5y) = 9x^2 - 25y^2$.
$(4pq + 7rst)(4pq - 7rst) = 16p^2q^2 - 49r^2s^2t^2$.

Note that we could not use this rule to multiply $3x + 5y$ by $3x - 5yz$, because these two quantities are not the sum and difference of the *same* pair of terms.

Reversing this rule, we are able to factorize at sight any binomial expression which consists of the difference of two squares.

Thus the factors of $x^2 - 16$ are $x + 4$ and $x - 4$; for x^2 is the square of x, and 16 is the square of 4. Thus—

$$x^2 - 16 = (x + 4)(x - 4)$$

Similarly: $4a^2 - 25b^2 = (2a + 5b)(2a - 5b)$

$$100x^2y^4z^6 - 9a^2b^2 = (10xy^2z^3 + 3ab)(10xy^2z^3 - 3ab)$$

EXAMPLE.—*Factorize* $20a^2x - 125b^2x$.

$$20a^2x - 125b^2x = 5x(4a^2 - 25b^2) \quad \ldots\ldots \quad \text{[Cf. § 88.}$$
$$= 5x(2a + 5b)(2a - 5b) \quad \text{[Difference of squares.}$$

EXAMPLE.—*Factorize completely* $x^4 - 16$.

$$x^4 - 16 = (x^2 + 4)(x^2 - 4)$$

But the second factor can be again factorized, for—

$$x^2 - 4 = (x + 2)(x - 2)$$

hence $x^4 - 16 = (x^2 + 4)(x + 2)(x - 2)$

EXAMPLES.—LXVI.

Write down the following products :—

1. $(p + q)(p - q)$. **2.** $(3a + 4b)(3a - 4b)$. **3.** $(5x^2 + 2y)(5x^2 - 2y)$.
4. $(3lm + 4hk)(3lm - 4hk)$. **5.** $(10 - xy^3z)(10 + xy^3z)$.
6. $(1 - 3abcd^2)(1 + 3abcd^2)$. **7.** $(\cdot 2 + xy)(\cdot 2 - xy)$.
8. $4(p + 3q)(p - 3q)$. **9.** $2ab(a + b)(a - b)$. **10.** $3x^2(2y + 7z)(2y - 7z)$.

Factorize—

11. $a^2 - 25b^2$.
12. $49h^2 - 16k^2$.
13. $9h^2k^2 - l^2m^2n^2$.
14. $121p^4 - 81q^2$.
15. $289x^2 - 225y^2z^2$.
16. $16x^8 - 9y^6$.
17. $12x^2y - 3y^3$.
18. $75a^2b^2c - 48c^3$.
19. $28abc^3 - 63a^3bca^2$.
20. $100a^2bc^2 - 25a^4b$.
21. $x^4 - 81$.
22. $x^8 - 256$.
23. $x^5 - 16x$.
24. $x^7 - 625x^3$.

90. On Trinomial Products.—It will frequently happen that the product of two binomial (*i.e.* two-termed) expressions is a Trinomial (*i.e.* three-termed expression).

ILLUSTRATIONS.—

$$\begin{array}{l} 3x - 2y \\ 4x + 3y \\ \hline 12x^2 - 8xy \\ \qquad + 9xy - 6y^2 \\ \hline 12x^2 + xy - 6y^2 \end{array} \qquad \begin{array}{l} 5x^2 - 2ab \\ 2x^2 + 7ab \\ \hline 10x^4 - 4abx^2 \\ \qquad + 35abx^2 - 14a^2b^2 \\ \hline 10x^4 + 31abx^2 - 14a^2b^2 \end{array}$$

Conversely, many trinomial expressions are the product of two binomial factors. To find these factors we must *guess* them, but we are guided by our knowledge of the type of multiplication given above.

Thus, if we consider the first of the above illustrations, note that—

(i.) The first term in the product (viz. $12x^2$) is the product of the first terms in the factors (viz. $3x$ and $4x$).

(ii.) The last term in the product (viz. $-6y^2$) is the product of the last terms in the factors (viz. $-2y$ and $+3y$).

EXAMPLE (1).—*Factorize* $x^2 + 5x + 6$.

Either of the following multiplication sums *might* give the required result :—

$$\begin{array}{c} x+1 \\ x+6 \\ \hline \end{array} \qquad \begin{array}{c} x-1 \\ x-6 \\ \hline \end{array} \qquad \begin{array}{c} x+2 \\ x+3 \\ \hline \end{array} \qquad \begin{array}{c} x-2 \\ x-3 \\ \hline \end{array}$$

For in each the first term of the product would be x^2, and the last term in the product would be $+6$.

It is easily seen, by multiplying out, that the third pair of factors is the only pair which will give $+5x$ as the middle term of the product. Thus—

$$x^2 + 5x + 6 = (x+2)(x+3)$$

Note that the second and fourth cases may be rejected in this example, as there are no minus signs in the required product.

EXAMPLE (2).—*Factorize* $x^2 - 3x - 28$.

Either of the following multiplication sums *might* give the required product :—

$$\begin{array}{c} x+28 \\ x-1 \\ \hline \end{array} \qquad \begin{array}{c} x-28 \\ x+1 \\ \hline \end{array} \qquad \begin{array}{c} x+4 \\ x-7 \\ \hline \end{array} \qquad \begin{array}{c} x-4 \\ x+7 \\ \hline \end{array} \qquad \begin{array}{c} x+14 \\ x-2 \\ \hline \end{array} \qquad \begin{array}{c} x-14 \\ x+2 \\ \hline \end{array}$$

For in each the first term of the product would be x^2, and the last term would be -28.

By actual multiplication we shall find that the third pair form the correct factors. Thus—

$$x^2 - 3x - 28 = (x - 7)(x + 4)$$

EXAMPLES.—LXVII.

Find the following products:—

1. $(x + 3)(x - 4)$. 2. $(x + 7)(x + 8)$. 3. $(x - 5)(x - 3)$.
4. $(x + 4y)(x + 5y)$. 5. $(x - 3z)(x + 5z)$. 6. $(x - yz)(x - 2yz)$.
7. $(x^2 + 4)(x^2 - 5)$. 8. $(x^2 - 3y^2)(x^2 - 7y^2)$.
9. $2y(x + 3y)(x - 6y)$. 10. $3x^2(x + 4y)(x - 9y)$.

Factorize—

11. $x^2 + 3x + 2$. 12. $x^2 + 3xy + 2y^2$. 13. $x^2 + 8x + 12$.
14. $x^2 + 8xy + 12y^2$. 15. $x^2 - 5x + 6$. 16. $x^2 - 3xy + 2y^2$.
17. $x^2 - 5x - 14$. 18. $a^2 + 5a - 24$. 19. $p^2 + pq - 20q^2$.
20. $p^2 - 7pq + 12q^2$. 21. $p^2 - 7pq - 30q^2$.
22. $3x^2y + 24xy + 36y$. 23. $5a^2b - 15ab^2 + 10b^3$.
24. $4p^3 + 20p^2 - 96p$. 25. $3x^2y^2 - 15x^2yz - 18x^2z^2$.
26. $x^2 - 16x + 60$. 27. $x^2 + 15x + 54$.
28. $p^2 - 10pq^2 - 24q^4$. 29. $h^2 + 12hk - 28k^2$.
30. $x^2y^2 + 20xyz + 36z^2$. 31. $a^2 + 8ab + 16b^2$.
32. $x^2 - 12xy + 36y^2$. 33. $p^2 - 10pq + 25q^2$.

91. EXAMPLE (1).—*Factorize* $2x^2 - 5xy - 3y^2$.

We try the following products:—

$2x + 3y$	$2x - 3y$	$2x + y$	$2x - y$
$x - y$	$x + y$	$x - 3y$	$x + 3y$

Each will give the first term $2x^2$, and the last term $-3y^2$; the third will be found to give the required result. Thus—

$$2x^2 - 5xy - 3y^2 = (2x + y)(x - 3y)$$

Note that in the above multiplications the first and second results differ only in the sign of the middle term; and the third and fourth results differ only in the same way.

Thus *if we change the sign of the* **second** *term in both factors, the result is to change the sign in the* **middle** *term of the product.* Remembering this, it would be necessary only to multiply out the first and third of these products. This will be shown in the next example.

EXAMPLE (2).—*Factorize* $6x^2 - 5xyz - 6y^2z^2$.

We try the following products:—

$6x - yz$	$6x - 6yz$	$6x - 3yz$	$6x - 2yz$	$3x - yz$	$3x - 6yz$	$3x - 2yz$	$3x - 3yz$
$x + 6yz$	$x + yz$	$x + 2yz$	$x + 3yz$	$2x + 6yz$	$2x + yz$	$2x + 3yz$	$2x + 2yz$

Each of these will give the first term and the last term in the required product. None of them will give the middle term $-5xyz$; but the seventh product will give the middle term $+5xyz$. Thus the seventh product will be correct, if we change the sign of the last term in each factor. Thus—

$$6x^2 - 5xyz - 6y^2z^2 = (3x + 2yz)(2x - 3yz)$$

Note also that we could reject the third product before multiplying it out, because $6x - 3zy$ is divisible by 3. But $6x^2 - 5yz - 6y^2z^2$ is not divisible by 3; therefore neither of its factors can be divisible by 3. For a similar reason, we could have rejected all the products in the above list except the first and seventh. This makes the work considerably less.

EXAMPLE (3).—*Factorize* $30x^2 + 68xy + 30y^2$.

$$30x^2 + 68xy + 30y^2 = 2(15x^2 + 34xy + 15y^2) \quad . \ . \ . \quad [\S\ 88.$$
$$= 2(5x + 3y)(3x + 5y)$$

using the method of the preceding examples.

EXAMPLES.—LXVIII.

Find the following products :—

1. $(2x + 5y)(4x + 3y)$. **2.** $(3x - 7)(4x + 5)$. **3.** $(7x - 2)(2x + 7)$.
4. $(6xy + 1)(3xy - 1)$. **5.** $(7a - bc)^2$. **6.** $3(2x-5)(5x+2)$.
7. $3xy(x + 2)(3x - 4)$.

Factorize—

8. $3x^2 + 11x - 4$. **9.** $3a^2 + 17a - 6$. **10.** $4p^2 - 21p + 5$.
11. $2x^2 + 7xy - 15y^2$. **12.** $15h^2 - 16h - 15$. **13.** $6p^2 + 13p + 6$.
14. $12p^2 + 25p + 12$. **15.** $15p^2 + 31p + 10$. **16.** $5a^2 + 12ab - 9b^2$.
17. $56c^2d^2 + cd - 1$. **18.** $21p^2q^2 - 4pqr - 32r^2$.
19. $12x^2 + 29xy + 14y^2$. **20.** $35a^2b^2 - 12ab + 1$.
21. $39 - 10yz - y^2z^2$. **22.** $18p^2 + 15pq - 12q^2$.
23. $12a^2x - 14abx - 10b^2x$. **24.** $15y^3z + 11y^2z^2 + 2yz^3$.
25. $210x^4 - 650x^3 + 500x^2$. **26.** $4a^2 - 12ab + 9b^2$.
27. $9a^2 + 6ab + b^2$. **28.** $9h^2 - 30hkl + 25k^2l^2$.
29. $49p^2 - 14p + 1$. **30.** $25 - 70xyz + 49x^2y^2z^2$.

92.—Multiplication and Division of Fractions.

EXAMPLE (1).—*Simplify* $\dfrac{a^2 - b^2}{a^2 + 4ab - 5b^2} \times \dfrac{a^2 + 10ab + (5b)^2}{a^2 + 4ab + 3b^2}$.

In multiplying these fractions we may cancel any **factor** in either numerator with the same **factor** if it occurs in either denominator. We therefore find the factors (if any) of each numerator and each denominator.

[Note particularly that we may not cancel a **term** from a numerator with the equal **term** from a denominator.]

After factorizing, we have—

$$\frac{(a - b)(a + b)}{(a - b)(a + 5b)} \times \frac{(a + 5b)(a + 5b)}{(a + b)(a + 3b)}$$

The factors $(a - b)$ cancel; also the factors $(a + b)$; and the factor $(a + 5b)$ in the first denominator cancels with one of the factors in the second numerator.

We are left with one factor in the numerator and one in the denominator, viz. $\dfrac{a + 5b}{a + 3b}$.

EXAMPLE (2).—*Simplify* $\dfrac{a^2 - b^2}{4x^2 - 9y^2} \times \dfrac{4x^2 - 12xy + 9y^2}{a^2 + 6ab + 5b^2} \div \dfrac{a + b}{a - b}$.

Factorizing, and inverting the last fraction—

$$\frac{(a - b)(a + b)}{(2x - 3y)(2x + 3y)} \times \frac{(2x-3y)(2x-3y)}{(a + b)(a + 5b)} \times \frac{a - b}{a + b}$$

Cancelling will remove one factor $(a+b)$, and one factor $(2x-3y)$, from both numerator and denominator. Multiplying the remaining factors, we obtain the result—

$$\frac{(a-b)^2(2x-3y)}{(2x+3y)(a+b)(a+5b)}$$

EXAMPLE (3).—*Reduce to its lowest terms* $\dfrac{3x^5y^2+12x^4y^3-15x^3y^4}{6x^6y-4x^5y^2-2x^4y^3}$.

$$3x^5y^2+12x^4y^3-15x^3y^4=3x^3y^2(x^2+4xy-5y^2)=3x^3y^2(x+5y)(x-y)$$
$$6x^6y-4x^5y^2-2x^4y^3=2x^4y(3x^2-2xy-y^2)=2x^4y(x-y)(3x+y)$$
$$\frac{3x^3y^2(x+5y)(x-y)}{2x^4y(x-y)(3x+y)}=\frac{3y(x+5y)}{2x(3x+y)}$$

For we remove the factors x^3, y, and $(x-y)$ from both numerator and denominator.

EXAMPLES.—LXIX.

Reduce to lowest terms—

1. $\dfrac{3x^2-15ax}{5ax-25a^2}$.
2. $\dfrac{8ax^2-12a^2x}{12x^3-18ax^2}$.
3. $\dfrac{10x^2-15ax}{14axy-21a^2y}$.
4. $\dfrac{12ab-15ac}{21ad+3ae}$.
5. $\dfrac{7p^3q+7pq^3}{21p^2q+42pq^2}$.
6. $\dfrac{36ab^2c+12abc^2}{42b^2cd-126bc^2d}$.
7. $\dfrac{a^2-b^2}{a^2+2ab+b^2}$.
8. $\dfrac{a^2-4ab+3b^2}{a^2-5ab+4b^2}$.
9. $\dfrac{4h^2-9k^2}{6h^2-13hk+6k^2}$.
10. $\dfrac{9p^2-12pq+4q^2}{15p^2-7pq-2q^2}$.
11. $\dfrac{48a^2c-75b^2c}{48a^2c+120abc+75b^2c}$.
12. $\dfrac{6x^3y-12x^2y^2-18xy^3}{15x^4-75x^3y+90x^2y^2}$.

Simplify—

13. $\dfrac{x+y}{x-y}\times\dfrac{3x^2-3xy}{5xy+5y^2}$.
14. $\dfrac{x^2+2xy+y^2}{2x-3y}\div\dfrac{2x^2-xy-3y^2}{x-y}$.
15. $\dfrac{6x^2-xy-12y^2}{2x^2+3xy-2y^2}\times\dfrac{x^2+xy-2y^2}{3x^2+xy-4y^2}\times(2x-y)$.
16. $\dfrac{4h^2-9k^2}{h^2-7hk+10k^2}\times\dfrac{h-2k}{2h-3k}\div\dfrac{2h^2+7hk+6k^2}{2h^2-9hk-5k^2}$.
17. $\dfrac{15c^2-3cd}{36cd+12d^2}\times\dfrac{15cd+5d^2}{6c^3-3c^2d}\times\dfrac{16c^2d-8cd^2}{5c^2+4cd-d^2}$.
18. $\dfrac{6x^3y-54xy^3}{10x^2z+80xyz+150y^2z}\times\dfrac{70xy^2z+154x^2yz+28x^3z}{42x^4-147x^3y+63x^2y^2}\times\dfrac{30x^2-35xy+10y^2}{16xy+8y^2}$.
19. $\dfrac{2x^2+5xy+2y^2}{6x^2-xy-2y^2}\div\left\{\dfrac{70xy-10y^2}{15x^2+10xy}\times\dfrac{3x^3+8x^2y+4xy^2}{42x^2y-34xy^2-4y^3}\right\}$.
20. $\dfrac{3x^3}{2y^2(x+2y)}\div\left\{\dfrac{30x^4-75x^3y+30x^2y^2}{4x^2y^3-16y^5}\div\left[\dfrac{4x^2-9y^2}{(x+y)^2}\times\dfrac{2x^2+xy-y^2}{2x+3y}\right]\right\}$.

93.—Addition and Subtraction of Fractions.—This follows the usual arithmetical method. To find the Lowest Common Denominators we must follow the

RULE.—**The Lowest Common Multiple of the denominators contains the highest power of every factor that occurs in any denominator.**

EXAMPLE (1). $\dfrac{3x}{x^2 - 5x + 6} + \dfrac{2x}{x^2 - 4}$.

Factorizing the denominators, we obtain—

$$\frac{3x}{(x-2)(x-3)} + \frac{2x}{(x+2)(x-2)}$$

Using the above rule, the Lowest Common Denominator is

$$(x+2)(x-2)(x-3)$$

Divide the L.C.D. by the first denominator—

$$(x+2)(x-2)(x-3) \div (x-2)(x-3) = \frac{(x+2)(x-2)(x-3)}{(x-2)(x-3)} = x+2$$

Multiplying this result by the first numerator, we obtain $3x(x+2)$, which is the first term in the required numerator.

Proceeding in the same way with the second fraction—

$$(x+2)(x-2)(x-3) \div (x+2)(x-2) = (x-3)$$
$$2x \times (x-3) = 2x(x-3)$$

which is the second term in the required numerator.

Thus we have—

$$\frac{3x}{(x-2)(x-3)} + \frac{2x}{(x+2)(x-2)} = \frac{3x(x+2) + 2x(x-3)}{(x+2)(x-2)(x-3)}$$

Multiplying out the numerator, we have—

$$3x^2 + 6x + 2x^2 - 6x \text{ ; } i.e. \ 5x^2$$

whence the result— $\dfrac{5x^2}{(x+2)(x-2)(x-3)}$

As in Arithmetic, most of the work is easily performed mentally.

EXAMPLE (2). $\dfrac{5(x+2)}{6x^2 - 13x + 6} + \dfrac{2(x-4)}{3x^2 + x - 2} - \dfrac{3(x+3)}{2x^2 - x - 3}$.

Factorizing the denominators—

$$\frac{5(x+2)}{(2x-3)(3x-2)} + \frac{2(x-4)}{(3x-2)(x+1)} - \frac{3(x+3)}{(x+1)(2x-3)}$$
$$= \frac{5(x+2)(x+1) + 2(x-4)(2x-3) - 3(x+3)(3x-2)}{(2x-3)(3x-2)(x+1)}$$

Simplifying the numerator by the method of § 70, we have—

$$-28x + 52 \text{ ; } i.e. \ 4(13 - 7x)$$

Thus we obtain the result—

$$\frac{4(13-7x)}{(2x-3)(3x-2)(x+1)}$$

Note that, as in Arithmetic, the answer should be reduced to its lowest terms. For this purpose we factorize the final numerator whenever possible, to see whether it has any factors in common with the denominator.

EXAMPLE (3). $\dfrac{3}{(2x-3)(x+3)} - \dfrac{4}{3(x-1)(x+3)} + \dfrac{1}{(2x-3)^2}$.

The denominators are already factorized.

Reducing to a common denominator by the above rule, we have—

$$\frac{9(2x-3)(x-1)-4(2x-3)^2+3(x-1)(x+3)}{3(2x-3)^2(x-1)(x+3)}$$

Simplifying the numerator by the method of § 70, we obtain—

$$5x^2+9x-18$$

but $$\frac{5x^2+9x-18}{3(2x-3)^2(x-1)(x+3)}=\frac{(5x-6)(x+3)}{3(2x-3)^2(x-1)(x+3)}$$

Reducing to lowest terms, we obtain $\dfrac{5x-6}{3(2x-3)^2(x-1)}$.

EXAMPLES.—LXX.

1. $\dfrac{1}{x+3y}-\dfrac{1}{2x+3y}$.

2. $\dfrac{3}{2x+4y}+\dfrac{2}{3x-6y}$.

3. $\dfrac{1}{3x-6y}+\dfrac{1}{3x+6y}-\dfrac{1}{x^2-4y^2}$.

4. $\dfrac{1}{a+2}+\dfrac{1}{a-2}-\dfrac{4}{a^2-4}$.

5. $\dfrac{2}{h+3k}-\dfrac{1}{h-3k}+\dfrac{9k}{h^2-9k^2}$.

6. $\dfrac{x+2y}{x-2y}-\dfrac{x-2y}{x+2y}$.

7. $\dfrac{6}{x^2+2x-8}-\dfrac{7}{x^2+x-12}+\dfrac{1}{x^2-5x+6}$.

8. $1+\dfrac{2x}{1-2x}$.

9. $1-\dfrac{2x}{1+2x}$.

10. $\left(1+\dfrac{2x}{1-2x}\right)\left(1-\dfrac{2x}{1+2x}\right)$.

11. $\dfrac{x}{a-y}-\dfrac{y}{a+2x}$.

12. $\dfrac{1}{x+y}+\dfrac{2}{y+z}$.

13. $p+q-\dfrac{9p^2-4q^2}{3p+2q}$.

14. $\dfrac{1}{2x+y}+\dfrac{1}{2x-y}-\dfrac{3x}{4x^2-y^2}$.

15. $\dfrac{1}{x^2-4x+3}-\dfrac{4}{x^2+2x-15}+\dfrac{3}{x^2+4x-5}$.

16. $\dfrac{x+3}{x+7}+\dfrac{4}{x-3}-\dfrac{40}{x^2+4x-21}$.

17. $\left\{\dfrac{1}{2a-2b}-\dfrac{1}{2a+2b}\right\}\div\left\{\dfrac{a}{a-b}-\dfrac{b}{a+b}\right\}$.

18. $\left\{\dfrac{x+y}{2x-2y}-\dfrac{x-y}{2x+2y}\right\}\times\left\{\dfrac{x}{2y}-\dfrac{y}{2x}\right\}$.

19. $\left\{\dfrac{1}{(a-b)^2}+\dfrac{1}{(a+b)^2}\right\}\div\left\{\dfrac{1}{(a-b)^2}-\dfrac{1}{(a+b)^2}\right\}$.

20. $\left\{\dfrac{1}{(p-q)(p^2-q^2)}+\dfrac{1}{(p+q)(p^2-q^2)}\right\}\div\left\{\dfrac{p}{(p-q)^2(p+q)}-\dfrac{q}{(p-q)(p+q)^2}\right\}$.

21. $\dfrac{1}{x^2+6x+5}-\dfrac{1}{2x^2+8x+6}+\dfrac{4}{x^2+10x+25}$.

22 $\dfrac{1}{4x^2-9x+5}+\dfrac{1}{5x^2-9x+4}-\dfrac{9}{(5x-4)^2}$.

94. Fractional Equations.

EXAMPLE. $\dfrac{2x+3}{2x-3} - \dfrac{3(x+2)}{3x-4} = \dfrac{16}{6x^2-17x+12}$

Factorizing denominators—

$$\frac{2x+3}{2x-3} - \frac{3(x+2)}{3x-4} = \frac{16}{(2x-3)(3x-4)}$$

Multiply both sides by the L.C.M. of the denominators, *i.e.* by $(2x-3)(3x-4)$; (cf. § 80)—

$$(2x+3)(3x-4) - 3(x+2)(2x-3) = 16$$
$$\textit{i.e.}\quad 6x^2 + x - 12 - 6x^2 - 3x + 18 = 16$$
$$\text{whence}\quad -2x = 10$$
$$\therefore x = -5$$

EXAMPLES.—LXXI.

1. $\dfrac{5}{x+10} + \dfrac{8}{x+4} = \dfrac{13}{x+7}$.

2. $\dfrac{x}{2+x} - \dfrac{10-3x}{9-3x} = \dfrac{5}{3-x}$.

3. $\dfrac{x+2}{x-4} + \dfrac{x+4}{x-2} = 2$.

4. $\dfrac{3}{x} + \dfrac{5}{x^2} = \dfrac{3}{x-1} + \dfrac{2}{(x-1)^2}$.

5. $\dfrac{a}{a-3} - \dfrac{a+2}{a-1} = \dfrac{a+3}{a-3} - \dfrac{a+6}{a-1}$.

6. $\dfrac{h+4}{h-3} + \dfrac{h+3}{h-4} = \dfrac{25}{h^2-7h+12}$.

7. $\dfrac{2p-15}{3p-12} + \dfrac{p-3}{3p+15} = \dfrac{p^2-5p}{p^2+p-20}$.

8. $3 + \dfrac{2a-3}{3a+1} = \dfrac{11a^2+3a+8}{3a^2-2a-1}$.

9. $\dfrac{1}{6x^2+7x-3} - \dfrac{4}{2x^2+7x+6} + \dfrac{3}{3x^2+5x-2} = 0$.

10. $\dfrac{4}{6x^2-11x-10} - \dfrac{5}{3x^2-7x-6} + \dfrac{1}{2x^2-11x+15} = 0$.

11. $\dfrac{2x-7}{3x+2} = \dfrac{2x-14}{3x-13}$.

12. $\dfrac{4a-1}{5a+1} = \dfrac{4a+2}{5a+7}$.

13. $\dfrac{2p-3}{3p-4} - \dfrac{2p-5}{3p-6} = \dfrac{1}{4p-8}$.

14. $y + 1 = \dfrac{4y^2}{4y+1}$.

15. $\dfrac{k}{1-5k} + 1 = \dfrac{9k^2-k}{1-6k+5k^2} + \dfrac{k}{1-k}$.

16. $\dfrac{2+p}{2p} + \dfrac{2p}{2+p} = \frac{5}{2}$.

CHAPTER XI.

ON SURDS AND INDICES.

95. On Products of Surds.—We have already given the definition of a surd in § 37, viz. that a surd is a root which does not work out exactly, *i.e.* which does not work out to any integer, or to a finite or recurring decimal.

RULE.—**To multiply a square root by itself, we omit the root sign.**

Or, expressing the rule as a formula—

$$\sqrt{x} \times \sqrt{x} = x$$
$$\text{or } (\sqrt{x})^2 = x$$

EXAMPLES (1).—$\sqrt{3} \times \sqrt{3} = 3$.

$$\begin{aligned} 2\sqrt{5} \times 6\sqrt{5} &= 2 \times \sqrt{5} \times 6 \times \sqrt{5} \\ &= 2 \times 6 \times \sqrt{5} \times \sqrt{5} \quad \text{(since the order of multiplication does not affect the result)} \\ &= 12 \times 5 \\ &= 60 \end{aligned}$$

$$\begin{aligned} a\sqrt{x} \times b\sqrt{x} &= a \times \sqrt{x} \times b \times \sqrt{x} \\ &= a \times b \times \sqrt{x} \times \sqrt{x} \quad \text{(since the order of multiplication does not affect the result)} \\ &= abx \end{aligned}$$

[Note that since $\sqrt{x} \times \sqrt{x} = x$, $\therefore x \div \sqrt{x} = \sqrt{x}$, thus $\frac{7}{\sqrt{7}} = \sqrt{7}$; $\frac{6}{\sqrt{2}} = \frac{3 \times 2}{\sqrt{2}} = 3\sqrt{2}$.]

EXPLANATION.—This follows at once from the definition, for the "square root of 3" means "that quantity which, when multiplied by itself, gives 3." That is to say, $\sqrt{3}$ *means a quantity such that* $\sqrt{3} \times \sqrt{3} = 3$.

[The corresponding statement—

$$\sqrt{25} \times \sqrt{25} = 25$$

is obvious, because $\sqrt{25}$ is 5.]

RULE.—**The product of the square roots of different quantities is the square root of the product of those quantities.**

Or, expressing the rule as a formula—

$$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$$

EXAMPLES (2).—(i.) $\sqrt{3} \times \sqrt{5} = \sqrt{15}$

(ii.) $2\sqrt{3} \times 7\sqrt{5} = 2 \times \sqrt{3} \times 7 \times \sqrt{5} = 2 \times 7 \times \sqrt{3} \times \sqrt{5} = 14\sqrt{15}$

(iii.) $3\sqrt{2} \times 5\sqrt{3} \times 2\sqrt{2} = 3 \times 5 \times 2 \times \sqrt{2} \times \sqrt{2} \times \sqrt{3}$
$= 3 \times 5 \times 2 \times 2 \times \sqrt{3} = 60\sqrt{3}$

(iv.) $p\sqrt{x} \times q\sqrt{y} = p \times \sqrt{x} \times q \times \sqrt{y} = p \times q \times \sqrt{x} \times \sqrt{y} = pq\sqrt{xy}$

(v.) $\sqrt{30} \div \sqrt{5} = \sqrt{6}$

(vi.) $30\sqrt{6} \div 5\sqrt{2} = \dfrac{30 \times \sqrt{6}}{5 \times \sqrt{2}} = 6\sqrt{3}$

(vii.) $a\sqrt{bc} \times b\sqrt{ca} \div c\sqrt{ab} = \dfrac{ab\sqrt{bc}\sqrt{ca}}{c\sqrt{ab}} = \dfrac{ab\sqrt{abc^2}}{c\sqrt{ab}} = \dfrac{ab\sqrt{c^2}}{c} = \dfrac{abc}{c} = ab$

EXPLANATION.—The law is obvious where the roots are *not* surds; for example—

$$\sqrt{25} \times \sqrt{4} = \sqrt{100}$$

for $\sqrt{25} = 5$, $\sqrt{4} = 2$, and $\sqrt{100} = 10$.

To prove that in all cases $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$, it is *only necessary to show that the square of* $\sqrt{x} \times \sqrt{y}$ *is always equal to the square of* $\sqrt{xy}$.

But the square of $\sqrt{x} \times \sqrt{y}$ is equal to—

$$\sqrt{x} \times \sqrt{y} \times \sqrt{x} \times \sqrt{y} = \sqrt{x} \times \sqrt{x} \times \sqrt{y} \times \sqrt{y} = xy;$$

while the square of $\sqrt{xy}$ is also equal to xy; which proves the law.

Similar rules hold for higher surds also. For instance, just as $\sqrt{x} \times \sqrt{x} = x$, so also—

$$\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x} = x, \text{ or } (\sqrt[3]{x})^3 = x$$

These statements follow from the definition of $\sqrt[3]{x}$.

Again, $(\sqrt[5]{x})^5 = x$; $(\sqrt[8]{x})^8 = x$; and so on.

Corresponding to the formula $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$, we have—

$$\sqrt[3]{x} \times \sqrt[3]{y} = \sqrt[3]{xy}; \quad \sqrt[7]{x} \times \sqrt[7]{y} = \sqrt[7]{xy}, \text{ etc.}$$

But surds *of different orders* may not be multiplied in this way; *e.g.* this rule does not enable us to multiply $\sqrt[3]{x}$ by $\sqrt[5]{y}$.

These two rules may be expressed thus—

$$\{\sqrt[n]{x}\}^n = x$$
$$\sqrt[n]{x} \times \sqrt[n]{y} = \sqrt[n]{xy}$$

EXAMPLES (3).—(i.) $2\sqrt[3]{5} \times 3\sqrt[3]{5} \times 4\sqrt[3]{5} = 2 \times 3 \times 4 \times \sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}$
$= 2 \times 3 \times 4 \times 5 = 120$

(ii.) $7\sqrt[3]{3} \times 8\sqrt[3]{4} = 56\sqrt[3]{12}$

(iii.) $2\sqrt[3]{3} \times 4\sqrt[4]{5} \times 6\sqrt[3]{7} \times 8\sqrt[4]{19} = 2 \times 4 \times 6 \times 8 \times \sqrt[3]{3} \times \sqrt[3]{7} \times \sqrt[4]{5} \times \sqrt[4]{19}$
$= 384\sqrt[3]{21}\sqrt[4]{95}$

(iv.) $3\sqrt[5]{4} \times 4\sqrt[5]{8} = 12\sqrt[5]{32} = 12 \times 2 = 24$

EXAMPLES.—LXXII.

Evaluate—

1. $3\sqrt{5} \times 2\sqrt{5}$. **2.** $10\sqrt{7} \times 3\sqrt{7}$. **3.** $3\sqrt{5} \times 4\sqrt{7}$.
4. $3\sqrt{2} \times 2\sqrt{3} \times 5\sqrt{5}$. **5.** $3\sqrt{2} \times 2\sqrt{3} \times 4\sqrt{2}$.
6. $7\sqrt{3} \times 2\sqrt{3} \times 4\sqrt{2}$. **7.** $2\sqrt{3} \times 3\sqrt{3} \times 4\sqrt{5}$.
8. $2\sqrt{3} \times 3\sqrt{3} \times 4\sqrt{3}$. **9.** $3\sqrt[3]{4} \times 8\sqrt[3]{3}$.
10. $5\sqrt[3]{7} \times 10\sqrt[3]{2}$. **11.** $10\sqrt[3]{4} \times 2\sqrt[3]{2}$.
12. $2\sqrt[3]{3} \times 4\sqrt[3]{3} \times 5\sqrt[3]{3}$. **13.** $8\sqrt[3]{5} \times \sqrt[3]{5} \times 2\sqrt[3]{5}$.
14. $2\sqrt[3]{2} \times 3\sqrt[3]{3} \times 4\sqrt[3]{4}$. **15.** $2\sqrt[3]{3} \times 3\sqrt[5]{5} \times 4\sqrt[3]{2} \times \sqrt[5]{6}$.
16. $3\sqrt[3]{2} \times 5\sqrt[5]{3} \times 4\sqrt[3]{2} \times \sqrt[5]{3}$ **17.** $4\sqrt{2} \times 5\sqrt[3]{3} \times 2\sqrt{3} \times 3\sqrt[3]{9}$.
18. $3\sqrt{2} \times 5\sqrt[5]{2} \times 4\sqrt{2} \times \sqrt[5]{4}$. **19.** $2\sqrt{3} \times 3\sqrt[3]{4} \times 4\sqrt{5} \times 6\sqrt[3]{2}$.
20. $2\sqrt[5]{2} \times 3\sqrt[5]{2} \times 4\sqrt[5]{8}$. **21.** $a\sqrt{b} \times b\sqrt{c} \times c\sqrt{bc}$.
22. $p\sqrt{qr} \times q\sqrt{rp} \times r\sqrt{pq}$. **23.** $a\sqrt{p} \times b\sqrt{q} \times c\sqrt{p}$.
24. $3a\sqrt{b} \times 4\sqrt{c} \times 2\sqrt{bc}$. **25.** $\sqrt[3]{a} \times b\sqrt[3]{c} \times d\sqrt[3]{e}$.
26. $p\sqrt[3]{q} \times q\sqrt[3]{q^2}$. **27.** $x\sqrt{y} \times y\sqrt{x} \times \sqrt[3]{x^2} \times \sqrt[3]{y}$.
28. $4a\sqrt{b} \times 3a\sqrt{c} \times \sqrt[3]{a} \times 2\sqrt[3]{a^2}$. **29.** $3\sqrt{6} \times 4\sqrt{15} \div 2\sqrt{10}$.
30. $4\sqrt{35} \times 5\sqrt{14} \div 2\sqrt{7}$. **31.** $3\sqrt{10} \times 4\sqrt{35} \div 3\sqrt{50}$.
32. $a\sqrt{b} \times b\sqrt{c} \div c\sqrt{bc}$. **33.** $p\sqrt{qr} \times q\sqrt{rp} \div r\sqrt{pq}$.
34. $a\sqrt{bc} \times b\sqrt{cd} \times c\sqrt{da} \div d\sqrt{a}$.

96. On Reduction of Surds.—By reversing a process of the last paragraph, we may factorize surds. Thus—

$$\sqrt{6} = \sqrt{3} \times \sqrt{2}$$
$$\sqrt[3]{10} = \sqrt[3]{2} \times \sqrt[3]{5}$$

This enables us to reduce many surds.

RULE.—**We can reduce a square root, if one factor of the radicand is a perfect square; a cube root, if one factor of the radicand is a perfect cube; and so on.**

EXAMPLES (i.)—
$$\sqrt{20} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$
$$\sqrt{1440} = \sqrt{144} \times \sqrt{10} = 12\sqrt{10}$$
$$\sqrt[3]{54} = \sqrt[3]{27} \times \sqrt[3]{2} = 3\sqrt[3]{2}$$
$$\sqrt[5]{64} = \sqrt[5]{32} \times \sqrt[5]{2} = 2\sqrt[5]{2}$$
$$3\sqrt{10} \times 4\sqrt{6} = 12\sqrt{60} = 12\sqrt{4} \times \sqrt{15} = 12 \times 2 \times \sqrt{15} = 24\sqrt{15}$$
$$3\sqrt{ab} \times 4\sqrt{bc} = 12\sqrt{ab^2c} = 12\sqrt{b^2} \times \sqrt{ac} = 12b\sqrt{ac}$$

It is perhaps worth while to remember the roots of a few numbers correct, at any rate, to four significant figures; they are given below correct to seven significant figures.

$$\sqrt{2} = 1{\cdot}414213\ldots \qquad \sqrt{3} = 1{\cdot}732050\ldots$$
$$\sqrt{5} = 2{\cdot}236068\ldots \qquad \sqrt{6} = 2{\cdot}449490\ldots$$
$$\sqrt{10} = 3{\cdot}162278\ldots$$

Using these values, we can work out many other surds without the trouble of extracting the square root.

EXAMPLES (ii.) $\sqrt{20} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5} = 2 \times 2{\cdot}236068 \ldots = 4{\cdot}472136$

$$3\sqrt{6} \times 5\sqrt{15} \div 2\sqrt{5} = \frac{3 \times 5}{2} \times \frac{\sqrt{6} \times \sqrt{15}}{\sqrt{5}} = \tfrac{15}{2} \times \sqrt{18} = \tfrac{15}{2} \times \sqrt{9}\sqrt{2}$$

$$= \tfrac{15}{2} \times 3\sqrt{2} = 31.82$$

$$\sqrt{1000} = \sqrt{100} \times \sqrt{10} = 10 \times 3{\cdot}162278 = 31{\cdot}62278$$

EXAMPLES.—LXXIII.

Reduce the following surds:—

1. $\sqrt{60}$. **2.** $\sqrt{28}$. **3.** $\sqrt{200}$. **4.** $\sqrt{98}$. **5.** $\sqrt{600}$.
6. $\sqrt{2000}$. **7.** $5\sqrt{20}$. **8.** $10\sqrt{8}$. **9.** $\sqrt[3]{81}$. **10.** $\sqrt[3]{48}$.
11. $\sqrt[3]{108}$. **12.** $\sqrt[3]{800}$. **13.** $\sqrt[5]{729}$. **14.** $\sqrt[5]{2430}$.

Find correct to five significant figures—

15. $\sqrt{12}$. **16.** $\sqrt{24}$. **17.** $\sqrt{50}$. **18.** $\sqrt{500}$. **19.** $4\sqrt{600}$.
20. $3\sqrt{48}$. **21.** $5\sqrt{40}$. **22.** $\sqrt{2312}$. **23.** $\sqrt{4500}$. **24.** $\sqrt{1728}$.
25. $2\sqrt{6} \times 5\sqrt{3}$. **26.** $3\sqrt{6} \times 5\sqrt{2}$.
27. $2\sqrt{15} \times 5\sqrt{3} \times 2\sqrt{2}$. **28.** $2\sqrt{15} \div 5\sqrt{3} \times 2\sqrt{2}$.
29. $24\sqrt{6} \times 3\sqrt{10} \div 2\sqrt{15}$. **30.** $2\sqrt{6} \times 3\sqrt{10} \times 2\sqrt{15} \div 3\sqrt{2}$.

97. On rationalizing Denominators.—Expressions which contain surds are called **irrational**; those which do not contain surds are called **rational.**

In evaluating a fraction whose denominator is irrational, we find it more convenient to obtain an equivalent fraction whose denominator is rational. This process is called "**rationalizing the denominator.**"

RULE.—**To rationalize the denominator of a fraction, if that denominator contains only one term, multiply both numerator and denominator by the surd contained in the denominator.**

EXAMPLES—

$$\frac{5}{\sqrt{3}} = \frac{5 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{5\sqrt{3}}{3}$$

$$\frac{10}{3\sqrt{2}} = \frac{10 \times \sqrt{2}}{3\sqrt{2} \times \sqrt{2}} = \frac{10\sqrt{2}}{6} = \frac{5\sqrt{2}}{3}$$

$$\frac{7}{3\sqrt{21}} = \frac{7 \times \sqrt{21}}{3\sqrt{21} \times \sqrt{21}} = \frac{7\sqrt{21}}{63} = \frac{\sqrt{21}}{9}$$

$$\frac{2\sqrt{3}}{5\sqrt{5}} = \frac{2\sqrt{3} \times \sqrt{5}}{5\sqrt{5} \times \sqrt{5}} = \frac{2\sqrt{15}}{25}$$

EXPLANATION.—By the laws of Arithmetic, since both numerator and denominator are multiplied by the same quantity, the value of the fraction is unaltered. Also, since the surd in the denominator becomes multiplied by itself, no root sign is left in the denominator. The object of this rule is *to make the fraction easier to evaluate.*

For instance, to evaluate $\frac{5}{\sqrt{3}}$, we must divide 5 by $1{\cdot}732050 \ldots$, which is a troublesome long-division sum. Whereas, if we rationalize the denominator, we obtain $\frac{5\sqrt{3}}{3} = \frac{5 \times 1{\cdot}732050}{3}$, which is very easily worked out.

EXAMPLES.—LXXIV.

Rationalize the denominators in the following fractions :—

1. $\frac{3}{4\sqrt{2}}$. 2. $\frac{5}{2\sqrt{3}}$. 3. $\frac{6\sqrt{2}}{5\sqrt{5}}$.

4. $\frac{2\sqrt{5}}{5\sqrt{3}}$. 5. $\frac{a\sqrt{b}}{c\sqrt{d}}$. 6. $\frac{ab\sqrt{cd}}{cd\sqrt{ab}}$.

Reduce to the equivalent decimal fraction, correct to five significant figures—

7. The fractions in Questions 1–4.

8. $\frac{10\sqrt{2}}{3\sqrt{3}}$. 9. $\frac{10\sqrt{5}}{7\sqrt{2}}$. 10. $\frac{125\sqrt{2}}{2\sqrt{5}}$.

11. $\frac{4}{\sqrt{3}} \times \frac{3}{\sqrt{5}}$. 12. $\frac{4\sqrt{2}}{3\sqrt{3}} \div \frac{2\sqrt{6}}{\sqrt{5}}$.

13. $\frac{3\sqrt{3}}{5\sqrt{5}} \times \frac{2\sqrt{3}}{5\sqrt{2}}$. 14. $\frac{7\sqrt{3}}{2\sqrt{5}} \div \frac{8\sqrt{6}}{3\sqrt{15}}$.

98. On the First Index Law.—We have already defined powers and indices (see § 33). It is now necessary to enter more fully into their properties. They are governed by three important rules, usually referred to as the **Three Index Laws.**

The First Index Law is as follows :—

To multiply powers of the same quantity, add their indices.

Or, expressing the law as a formula—

$$a^m \times a^n = a^{m+n}$$

This law has already been explained in § 60, and in § 64 we have stated the same law in the form for division—

To divide two powers of the same letter, subtract the index of the divisor from that of the dividend.

Or, expressed as a formula—

$$a^m \div a^n = a^{m-n}$$

This law is so important that we will repeat the explanation of it.

Let us find the product of a^5 and a^3. Since a^5 means $a \times a \times a \times a \times a$ (*i.e.* the continued product of five a's), and a^3 means $a \times a \times a$ (the continued product of three a's) ; therefore $a^5 \times a^3$ gives $(a \times a \times a \times a \times a) \times (a \times a \times a)$, which is the continued product of eight a's, and is therefore represented by a^8.

Thus we have $a^5 \times a^3 = a^8$, from which it follows that $a^8 \div a^3 = a^5$.

These two statements are examples of the law in each form.

Note the extension of the law, $a^m \times a^n \times a^p = a^{m+n+p}$, etc.

99. The Second Index Law.—To find a power of a power, multiply the indices. Thus the fourth power of the cube of a is the twelfth power of a ; for $4 \times 3 = 12$.

Expressing the law as a formula—

$$(a^m)^n = a^{mn},$$

(The nth power of the mth power of a is the mnth power of a.)

EXPLANATION.—Let us find the value of the cube of a^5.

$$(a^5)^3 = a^5 \times a^5 \times a^5$$

and since the multiplication is performed by *adding the indices*, the result is a^{15}; and this agrees with the law given.

Note the extension of the law—

$$\{(a^m)^n\}^p = a^{mnp}\text{; etc.}$$

100. The Third Index Law.—To raise a product to any power, raise each factor to that power. For example, the cube of abc is $a^3b^3c^3$.

Expressing the law as a formula—

$$(abc)^n = a^nb^nc^n$$

The same law applies to a quotient, thus—

$$(a \div b)^n = a^n \div b^n$$

$$\text{or} \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Combining these two forms of the law, we obtain the more complete formula—

$$\left(\frac{ab}{cd}\right)^n = \frac{a^nb^n}{c^nd^n}$$

This law is often called the **Distributive Law,** because the index is "*distributed*" to each factor of both numerator and denominator.

EXPLANATION.—Let us find the value of $\left(\frac{ab}{cd}\right)^4$

$$\left(\frac{ab}{cd}\right)^4 = \frac{ab}{cd} \times \frac{ab}{cd} \times \frac{ab}{cd} \times \frac{ab}{cd}$$

$$= \frac{a \times b \times a \times b \times a \times b \times a \times b}{c \times d \times c \times d \times c \times d \times c \times d}$$

$$= \frac{a \times a \times a \times a \times b \times b \times b \times b}{c \times c \times c \times c \times d \times d \times d \times d}$$

$$= \frac{a^4b^4}{c^4d^4}$$

Caution.—The student must be very careful to notice that the Distributive Law for Indices *does not apply* to a series of terms, but only to a series of factors and divisors. Thus—

$$\left(\frac{ab}{cd}\right)^2 = \frac{a^2b^2}{c^2d^2}$$

but $(a + b)^2$ is *not* equal to $a^2 + b^2$, but to $a^2 + 2ab + b^2$ (see § 75).

EXAMPLES.—

(1) $a^3 \times a^4 \times a^5 = a^{3+4+5} = a^{12}$. [by Law I.

(2) $a^5 \times a^4 \div a^6 \times a = a^{5+4-6+1} = a^4$. [by Law I.

(3) $a^2b^3 \times a^4b^4 \div ab^2 = a^{2+4-1}b^{3+4-2} = a^5b^5$. [by Law I.

(4) $3x^2y^3 \times 4x^5y^5 \div 6xy^2 = 2x^{2+5-1}y^{3+5-2} = 2x^6y^6$. [by Law I.

(5) $px^yy^x \times qx^xy^y = pqx^{y+x}y^{x+y}$. [by Law I.

(6) $(p^6)^7 = p^{42}$; $(q^r)^s = q^{rs}$. [by Law II.

(7) $(a^2b^3)^4 = (a^2)^4(b^3)^4$ [by Law III.

$= a^8b^{12}$. [by Law II.

(8) $\left(\frac{3x^2y^4}{4p^5q^6}\right)^3 = \frac{3^3(x^2)^3(y^4)^3}{4^3(p^5)^3(q^6)^3}$. [by Law III.

$= \frac{27x^6y^{12}}{64p^{15}q^{18}}$. [by Law II.

(9) $(x^2y^3)^4 \div (x^3y^2)^5 = \frac{(x^2)^4(y^3)^4}{(x^3)^5(y^2)^5}$ [by Law III.

$= \frac{x^8y^{12}}{x^{15}y^{10}}$ [by Law II.

$= \frac{y^2}{x^7}$. [cancelling by Law I.

EXAMPLES.—LXXV.

Evaluate—

1. $3a^3 \times 5a^2 \times 2a^4$; $2x^5 \times 3x^3 \times 5x^2$; $2x \times 3x^2 \times x^3$.

2. $5a^5 \times 4a^4 \div 2a^2$; $10p^3 \times 3p^5 \div 6p$; $8p^3 \times 2p^2 \div p^4$.

3. $3ab \times 4a^2b^2 \times 5a^3b^4$; $6pq \times 3p^2q^3 \times 2p^5q^3$; $2x^2y^3z \times 5xy^5z \times 6x^4y^2z^3$.

4. $4a^3b^2c^2 \times 2a^3bc^3 \div 4a^2bc$; $5p^2q^3r \times 14p^3q^5r^2 \div 7p^2qr$; $4x^3y^2z \times 9x^2y^4z^5 \div 6x^3y^3z^3$.

5. $3a^2b^3 \times 4b^2c^3 \div 6ac^2$; $7p^5q^5 \times 8q^3r^5 \div 14p^3r^3$; $10a^5 \times 11b^5c^6 \times 3a^2c^4 \div 66a^3b^2c^5$.

6. $(3a^3b^3)^4$; $(4a^2b^2)^3$; $(2a^2b^2)^5$; $(3xy^2z^3)^4$.

7. $(a^2b^3)^4 \times (3a^3b^2)^2$; $2a^4b^3 \times (3ab^2)^2$; $(4a^2b^3)^4 \times (2a^3b^2)^2$.

8. $(2a^2b^2)^3 \times (3a^3b^3)^2 \div (6a^2b^2)^2$; $(3a^3b^3)^4 \times (2a^2b^2)^3 \div (6a^3b^3)^3$; $(2ab^2)^5 \times (a^4b^3)^2 \div (a^3b^4)^3$.

9. $\left(\frac{3a^2b^2}{cd^3}\right)^2 \times \left(\frac{2a^3c^3}{b^3d^2}\right)^2 \div \left(\frac{2a^4}{d^5}\right)^3$; $\frac{2x^2y^3}{z^3} \times \left(\frac{3y^3z^2}{x}\right)^3 \div \left(\frac{4y^3}{z^4x^3}\right)^2$.

101. Fractional, Zero, and Negative Indices.—The next step in the line of our work is to assign a meaning to indices *which are not integers*. We know the meaning of the symbol a^4; but the symbols $a^{\frac{2}{3}}$, a^0, a^{-4}, have as yet no meaning to us.

In § 98, we have proved three important laws which govern indices, *provided these indices are positive integers*; the meanings which we assign to fractional, zero, and negative indices are such that *all indices obey these same laws.*

The full consideration of these indices is much beyond the scope of this book; but an intelligent use of them is very desirable, and for this purpose it will be well to study carefully the following explanation.

(i.) **The Zero Index.**—When any quantity is divided by itself, the quotient is 1. Thus $a^5 \div a^5 = 1$.

But if we suppose the First Index Law to be applicable to this case, we have $a^5 \div a^5 = a^{5-5} = a^0$.

Hence, to be consistent, we must suppose that a^0 *means* 1. By a similar argument, the zero power of *any quantity* is unity.

(ii.) **The Positive Fractional Index.**—*To find a meaning for* $a^{\frac{1}{3}}$. Applying the Second Index Law—

$$(a^{\frac{1}{3}})^3 = a^{\frac{1}{3}\times 3} = a^1 = a$$

Thus, *if we cube* $a^{\frac{1}{3}}$ *we get* a; therefore $a^{\frac{1}{3}}$ means the *cube root of* a. (See the definition of cube root, § 36.)

To find a meaning for $a^{\frac{3}{5}}$.

If we apply the Second Index Law—

$$(a^{\frac{3}{5}})^5 = a^{\frac{3}{5}\times 5} = a^3$$

Thus, the fifth power of $a^{\frac{3}{5}}$ is a^3, hence $a^{\frac{3}{5}}$ is the fifth root of a^3; *i e.*

$$a^{\frac{3}{5}} = \sqrt[5]{a^3}$$

Similar reasoning would suggest that—

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

(iii.) **The Negative Index.**—*To find a meaning for* a^{-2}.

$a^3 \div a^5 = \frac{a^3}{a^5}$; and if we reduce this fraction to lowest terms by dividing both numerator and denominator by a^3, we obtain as result $\frac{1}{a^2}$; *i.e.*

$$a^3 \div a^5 = \frac{1}{a^2}$$

But if we apply the First Index Law—

$$a^3 \div a^5 = a^{3-5} = a^{-2}$$

Hence, to be consistent, we must suppose that a^{-2} *means* $\frac{1}{a^2}$.

Similar reasoning would suggest that in all cases—

$$a^{-n} \text{ means } \frac{1}{a^n}$$

EXAMPLES.—LXXVI.

1. The following statements are applications of the First Index Law: show that they are consistent with the hypothesis that the zero power of any quantity is unity :—

$$a^3 \times a^0 = a^3;\ \ x^4 \div x^4 = x^0;\ \ a^n \times a^0 = a^n$$

2. Show that the following statements (which are derived from the Index Laws) are consistent with the hypothesis that $a^{\frac{p}{q}}$ means $\sqrt[q]{a^p}$:—

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1;\ a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^2;\ (x^{\frac{3}{2}})^2 = x^3;\ (a^{\frac{1}{4}})^2 = a^{\frac{1}{2}};\ (x^{\frac{1}{2}})^4 = x^2.$$

3. Show that the following statements (which are derived from the Index Laws) are consistent with the hypothesis that a^{-n} means $\frac{1}{a^n}$:—

$a^2 \div a^5 = a^{-3}$; $x^5 \times x^{-2} = x^3$; $x^{-2} \times x^{-3} = x^{-5}$; $b^5 \div b^{-2} = b^7$; $p^3 \times p^{-3} = p^0$.

102. Without further discussion, we will now accept the following definitions, and also assume that the three Index Laws may be applied to all indices and to every combination of indices :—

$a^{\frac{p}{q}}$ is used to represent $\sqrt[q]{a^p}$
a^0 ,, ,, 1
a^{-n} ,, ,, $\frac{1}{a^n}$

Illustrations.—$a^{\frac{2}{3}}$ means $\sqrt[3]{a^2}$; $p^{\frac{5}{8}}$ means $\sqrt[8]{p^5}$; $q^{\frac{1}{10}}$ means $\sqrt[10]{q}$.

Any quantity to the zero power means 1; thus, x^0 means 1; m^0 means 1; $(\cdot 35)^0$ means 1.

a^{-3} means $\frac{1}{a^3}$; x^{-7} means $\frac{1}{x^7}$; $p^{-\frac{2}{3}}$ means $\frac{1}{p^{\frac{2}{3}}}$, *i.e.* $\frac{1}{\sqrt[3]{p^2}}$.

EXAMPLES (1).—*Evaluate* $4^{\frac{3}{2}}$, $9^{\frac{1}{4}}$, $25^{\frac{3}{2}}$, $8^{\frac{2}{3}}$, $27^{\frac{1}{3}}$, $125^{\frac{5}{3}}$.

$4^{\frac{3}{2}} = \sqrt[2]{4^3} = \sqrt{64} = 8$.

$9^{\frac{1}{4}} = \sqrt[4]{9}$; but the fourth root is the square root of the square root; thus, since the square root of 9 is 3, the fourth root of 9 is $\sqrt{3}$.

$25^{\frac{3}{2}} = \sqrt[2]{25^3}$: this means that we are to cube 25, and then extract the square root of the result; it will be found that we obtain the same answer, with far less trouble, if we reverse the order of the operations, *i.e.* (i.) extract the square root of 25, and (ii.) cube the result. Thus $\sqrt{25} = 5$; $5^3 = 125$.

$8^{\frac{2}{3}}$: following the same rule, $\sqrt[3]{8} = 2$; $2^2 = 4$.

$27^{\frac{1}{3}} = \sqrt[3]{27} = 3$.

$125^{\frac{5}{3}}$: $\sqrt[3]{125} = 5$; $5^5 = 3125$.

EXAMPLES (2).—*Evaluate* 8^{-2}, 10^{-3}, $8^{-\frac{2}{3}}$, $100^{-\frac{3}{2}}$, $(-64)^{-\frac{1}{3}}$, $(-25)^{-2}$, $(-8)^{\frac{4}{3}}$.

$$8^{-2} = \frac{1}{8^2} = \frac{1}{64}.$$

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1000}.$$

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{4}.$$

$$100^{-\frac{3}{2}} = \frac{1}{100^{\frac{3}{2}}} = \frac{1}{1000}.$$

$$(-64)^{-\frac{1}{3}} = \frac{1}{(-64)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{-64}} = \frac{1}{-4} = -\frac{1}{4}.$$

$$(-25)^{-2} = \frac{1}{(-25)^2} = \frac{1}{625}.$$

$$(-8)^{\frac{4}{3}} = \sqrt[3]{(-8)^4} = (-2)^4 = 16.$$

EXAMPLES (3).—*Evaluate* $4^{1\cdot 5}$, $9^{-2\cdot 5}$, $16^{\cdot 5}$, $27^{\cdot\dot{3}}$, $(-8)^{\cdot\dot{3}}$.

$$4^{1\cdot 5} = 4^{\frac{3}{2}} = \sqrt{4^3} = 8.$$

$$9^{-2\cdot 5} = 9^{-\frac{5}{2}} = \frac{1}{9^{\frac{5}{2}}} = \frac{1}{\sqrt{9^5}} = \frac{1}{243}.$$

$$16^{\cdot 5} = 16^{\frac{1}{2}} = \sqrt{16} = 4.$$

$$27^{\cdot\dot{3}} = 27^{\frac{1}{3}} = \sqrt[3]{27} = 3.$$

$$(-8)^{\cdot\dot{3}} = (-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2.$$

EXAMPLES.—LXXVII.

Evaluate—

1. $4^{\frac{1}{2}}$, $25^{\frac{1}{2}}$, $8^{\frac{1}{3}}$, $64^{\frac{1}{3}}$, $16^{\frac{1}{4}}$, $25^{\frac{1}{4}}$, $64^{\frac{1}{6}}$.
2. $4^{\frac{3}{2}}$, $25^{\frac{5}{2}}$, $8^{\frac{2}{3}}$, $64^{\frac{4}{3}}$, $16^{\frac{3}{4}}$, $25^{\frac{3}{4}}$, $64^{\frac{5}{6}}$.
3. $(-125)^{\frac{1}{3}}$, $(-8)^{\frac{2}{3}}$, $(-27)^{\frac{4}{3}}$, $(-32)^{\frac{1}{5}}$, $(-343)^{\frac{2}{3}}$.
4. 2^{-3}, 3^{-2}, 5^{-2}, 2^{-4}, 3^{-5}.
5. $25^{-\frac{1}{2}}$, $64^{-\frac{1}{3}}$, $25^{-\frac{1}{4}}$, $4^{-\frac{3}{2}}$, $8^{-\frac{2}{3}}$, $16^{-\frac{3}{4}}$, $64^{-\frac{5}{6}}$.
6. $9^{1\cdot 5}$, $4^{2\cdot 5}$, $81^{\cdot 75}$, $16^{\cdot 25}$.
7. $16^{-\cdot 5}$, $9^{-\cdot 25}$, 3^0, $49^{-1\cdot 5}$, $25^{-2\cdot 5}$, 49^0, $(-64)^{-\cdot\dot{3}}$
8. $\left(\frac{9}{25}\right)^{\frac{1}{2}}$, $\left(\frac{81}{625}\right)^{\frac{3}{4}}$, $\left(-\frac{1}{125}\right)^{\frac{2}{3}}$, $\left(\frac{25}{64}\right)^{\frac{3}{2}}$, $\left(\frac{81}{49}\right)^{\frac{5}{2}}$.
9. $(\cdot 5)^{-2}$, $(\cdot 5)^{-3}$, $(\cdot 2)^{-4}$, $(-\cdot 3)^{-3}$, $(\cdot 12)^{-1}$, $(\cdot 23)^0$.
10. $(\cdot 25)^{\frac{1}{2}}$, $(\cdot 25)^{\frac{3}{2}}$, $(\cdot 125)^{\frac{2}{3}}$, $(-\cdot 125)^{\frac{4}{3}}$, $(\cdot 0625)^{\frac{3}{2}}$.
11. $(\cdot 25)^{-\frac{1}{2}}$, $(\cdot 125)^{-\frac{2}{3}}$, $(-\cdot 008)^{-\frac{4}{3}}$, $(\cdot 216)^{\frac{2}{3}}$, $(\cdot 64)^{-\frac{3}{2}}$.
12. $(6\cdot 25)^{\frac{1}{2}}$, $(2\cdot 25)^{-\frac{3}{2}}$, $(-\cdot 729)^{-\frac{2}{3}}$, $(1\cdot 44)^{-\frac{3}{2}}$, $(\cdot 0144)^{-\frac{1}{2}}$.

103. Application of the Index Laws.

EXAMPLE (1).—*Use the Index Laws to simplify* $\sqrt[3]{2} \times \sqrt[5]{2} \times \sqrt{2} \div \sqrt[30]{2}$.

$\sqrt[3]{2}$ may be expressed as $2^{\frac{1}{3}}$, etc. Thus—

$$\sqrt[3]{2} \times \sqrt[5]{2} \times \sqrt{2} \div \sqrt[30]{2} = 2^{\frac{1}{3}} \times 2^{\frac{1}{5}} \times 2^{\frac{1}{2}} \div 2^{\frac{1}{30}}$$

$$= 2^{\frac{1}{3}+\frac{1}{5}+\frac{1}{2}-\frac{1}{30}} = 2^1 = 2$$

EXAMPLE (2).—*Simplify* $\sqrt[3]{a^2} \times \sqrt{a^3} \times (\sqrt{a})^{-\frac{1}{3}}$.

Writing the surds as powers, we obtain—

$$a^{\frac{2}{3}} \times a^{\frac{3}{2}} \times (a^{\frac{1}{2}})^{-\frac{1}{3}} = a^{\frac{2}{3}} \times a^{\frac{3}{2}} \times a^{-\frac{1}{6}} \quad \text{[by Law II.}$$

$$= a^{\frac{2}{3}+\frac{3}{2}-\frac{1}{6}} \quad \text{[by Law I.}$$

$$= a^2$$

EXAMPLE (3).—*Simplify* $(a^{\frac{1}{2}}b^{\frac{3}{2}})^{-2} \times \left(\frac{a^2b}{c^2d}\right)^{\frac{1}{3}} \times \left(\frac{ac^2}{bd^2}\right)^{-\frac{2}{3}}$

This gives $a^{-1}b^{-3} \times \dfrac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{c^{\frac{2}{3}}d^{\frac{1}{3}}} \times \dfrac{a^{-\frac{2}{3}}c^{-\frac{4}{3}}}{b^{-\frac{2}{3}}d^{-\frac{4}{3}}}$ [by Laws III. and II.

$= a^{-1+\frac{2}{3}-\frac{2}{3}}b^{-3+\frac{1}{3}+\frac{2}{3}}c^{-\frac{2}{3}-\frac{4}{3}}d^{-\frac{1}{3}+\frac{4}{3}}$ [by Law I.

$= a^{-1}b^{-2}c^{-2}d^{1}$

$= \frac{1}{a}\cdot\frac{1}{b^2}\cdot\frac{1}{c^2}\cdot d$ [by Definition.

$= \dfrac{d}{ab^2c^2}$

EXAMPLE (4).—*Simplify* $\sqrt{2} \times \sqrt[3]{9} \times \sqrt[4]{64} \div \sqrt[6]{729}$.

This gives $\sqrt{2} \times \sqrt[3]{3^2} \times \sqrt[4]{2^6} \div 3$

$= 2^{\frac{1}{2}} \times 3^{\frac{2}{3}} \times 2^{\frac{3}{2}} \div 3$ [by Definition.

$= 2^{\frac{1}{2}-\frac{3}{2}} \times 3^{\frac{2}{3}-1}$ [by Law I.

$= 2^{-1} \times 3^{-\frac{1}{3}}$

$= \frac{1}{2} \times \dfrac{1}{\sqrt[3]{3}}$ [by Definition.

$= \dfrac{1}{2\sqrt[3]{3}}$

EXAMPLES.—LXXVIII.

Simplify—

1. $\sqrt{2} \times \sqrt[3]{2} \times \sqrt[4]{2} \div \sqrt[12]{2}$.
2. $\sqrt[3]{3} \times \sqrt[4]{3} \times \sqrt[5]{3} \div \sqrt[30]{3}$.
3. $\sqrt[3]{a^2} \div \sqrt[4]{a^3} \times \sqrt[5]{a^4}$.
4. $\sqrt[3]{x^2} \div \sqrt[5]{x^4} \times \sqrt[7]{x^6}$.
5. $(a^{\frac{1}{3}})^{\frac{3}{2}} \times (a^{-\frac{1}{3}})^{-\frac{3}{2}} \times (a^{\frac{2}{3}})^{-\frac{3}{2}}$.
6. $(a^2)^{-\frac{1}{2}} \times (a^{\frac{1}{2}})^{-2} \times a^2$.
7. $\sqrt[3]{4} \times \sqrt[4]{8} \times \sqrt[3]{16} \times \sqrt[8]{4}$.
8. $\sqrt[5]{4} \times \sqrt[4]{9} \times \sqrt[5]{8}$.
9. $\sqrt[3]{25} \times \sqrt[3]{2} \times \sqrt[3]{625} \div \sqrt[3]{4}$.
10. $\left(\frac{ab^2}{cd^3}\right)^{\frac{1}{2}} \times \left(\frac{a^2b}{c^3d}\right)^{-\frac{1}{2}} \div \left(\frac{ab}{cd}\right)^{-\frac{1}{2}}$.
11. $(abcd)^{\frac{1}{3}} \times (a^2b^2c^2d^2)^{-\frac{1}{3}} \div \left(\frac{a^3b^3}{c^3d^3}\right)^{\frac{1}{9}}$.
12. $(pqr)^{\frac{1}{2}} \times (qrs)^{-\frac{1}{3}} \times (rsp)^{\frac{1}{4}} \div (spq)^{\frac{1}{6}}$.
13. $(xy)^{\frac{2}{3}} \times (yz)^{-\frac{3}{4}} \times (zx)^{\frac{5}{6}} \times (xyz)^{-\frac{1}{6}}$.
14. $\sqrt{xy} \div \sqrt[3]{y^2z^2} \times \sqrt[4]{z^3x^3} \div \sqrt[6]{xyz}$.
15. $\sqrt{a^2b^3} \div \sqrt[3]{b^4c^6} \times \sqrt[4]{c^6a^8} \div \sqrt[6]{a^6b^2c^3}$.

104. On Powers of 10.—It is obvious that $1000 = 10^3$, $100{,}000{,}000 = 10^8$, $\cdot001 = 10^{-3}$, $\cdot000000001 = 10^{-9}$; and in general that any **quantity which has 1 for its only significant figure can be expressed as a power of 10.**

Again—

$$32{\cdot}48 = 3{\cdot}248 \times 10 \qquad {\cdot}3248 = 3{\cdot}248 \times 10^{-1}$$
$$324{\cdot}8 = 3{\cdot}248 \times 10^2 \qquad {\cdot}03248 = 3{\cdot}248 \times 10^{-2}$$
$$3248 = 3{\cdot}248 \times 10^3 \qquad {\cdot}003248 = 3{\cdot}248 \times 10^{-3}$$
$$32480 = 3{\cdot}248 \times 10^4 \qquad {\cdot}0003248 = 3{\cdot}248 \times 10^{-4}$$
$$324800 = 3{\cdot}248 \times 10^5 \qquad {\cdot}00003248 = 3{\cdot}248 \times 10^{-5}$$
etc. etc.

Any other set of figures may be treated in the same way; thus it is easily seen that **any quantity which does not lie between 1 and 10 can be expressed as a quantity which does lie between 1 and 10 multiplied by some positive or negative power of 10.**

This notation is exceedingly useful for many purposes, especially in dealing with very large or very small quantities. It is far more concise than the ordinary notation, and is soon found to be more suggestive to the mind.

EXAMPLES.—*Express the following numbers by the method described above:*—

(1) 32874000. Move the decimal point **7** places to the *left*, and multiply by 10^7. Result, $3{\cdot}2874 \times 10^7$.

(2) ·000028415. Move the decimal point **5** places to the *right*, and multiply by 10^{-5}. Result, $2{\cdot}8415 \times 10^{-5}$.

(3) Evaluate $32400000 \times 1120000 \div {\cdot}0000000504$

$$(3{\cdot}24 \times 10^7) \times (1{\cdot}12 \times 10^6) \div (5{\cdot}04 \times 10^{-8})$$
$$= \frac{3{\cdot}24 \times 1{\cdot}12 \times 10^7 \times 10^6}{5{\cdot}04 \times 10^{-8}} = {\cdot}72 \times 10^{13} \div 10^{-8}$$
$$= 7{\cdot}2 \times 10^{-1} \times 10^{13} \div 10^{-8} = 7{\cdot}2 \times 10^{20}$$

EXAMPLES.—LXXIX.

Express the following quantities by the notation of the preceding paragraph:—

1. 30,507,000. **2.** 2,499,300,000. **3.** ·0000007891.
4. 368·21. **5.** Seventy-three million million.
6. ·0000000000002854. **7.** 875,000,000 × 12,800,000.
8. ·000000875 × ·00000128.
9. ·0000484 × ·000000112 ÷ ·00000001232.
10. ·000000875 × 125,000,000.

Work the following correct to three significant figures:—

11. 827,000,000 × 12,900,000 ÷ ·000000536.
12. 215 × ·000000827 ÷ ·000000000234.
13. ·372 × ·00483 ÷ 92,400,000.
14. 2,380,000 × 91,200,000 ÷ ·000000576.

CHAPTER XII.

ON VARIATION.

105. THE mathematical theory of **Variation** is in reality the arithmetical theory of proportion in a more scientific form, and considerably extended.

It deals with all cases where the magnitude of one quantity depends on the magnitude of some other quantity or quantities. As examples of such cases we may quote the following :—

The *weight* of a piece of gold depends on its *volume ;*

The *time* required for a journey depends on the *distance travelled* and the *rate of travelling* ;

The *volume* of a sphere depends on its *diameter ;*

The *pressure* of a gas depends on its *density* and its *temperature.*

106. Direct Variation—Definition.—One quantity is said to **vary directly as another quantity, if the ratio of any two values of the first quantity is equal to the ratio of the corresponding two values of the second quantity.**

The symbol $\propto$ is used to denote direct variation ; thus, "$L \propto M$" means "L varies directly as M."

Illustration of direct variation.—If we have two lumps of gold whose volumes are in the ratio 2 : 3, their weights will also be in the ratio 2 : 3 ; or, if their volumes are in the ratio 5 : 7, their weights will also be in the ratio 5 : 7 ; etc.

Thus *the weight of a piece of gold varies directly as its volume.*

EXAMPLE (1).—*If* P *varies directly as* Q, *and if* P *is* 20 *when* Q *is* 36, *find* P *when* Q *is* 45.

Let x be the required value of P ; then, by definition, the ratio of the second value of P to the first value of P is equal to the ratio of the corresponding values of Q.

$$i.e. \quad \frac{x}{20} = \frac{45}{36} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{[See § 19.}$$

$$\text{or} \quad \frac{x}{20} = \frac{5}{4}$$

$$\text{whence} \quad x = 25 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{[See § 80.}$$

EXAMPLE (2).—*The circumference of a circle varies directly as its diameter ; and when the diameter is* 21 *inches the circumference is* 66 *inches : find the circumference when the diameter is* 35 *inches.*

The ratio of the second circumference to the first circumference is equal to the ratio of the second diameter to the first diameter.

$$i.e.\quad \frac{x}{66} = \frac{35}{21}$$

$$i.e.\quad \frac{x}{66} = \frac{5}{3}$$

$$\therefore\quad x = 110 \text{ inches}$$

EXAMPLE (3).—*The weight of a sphere varies directly as the cube of its diameter; when the diameter is* 5 *cms. the weight is* 500 *gms.: find the diameter when the weight is* 1372 *gms.*

Cube of second diameter : cube of first diameter

= weight of second sphere : weight of first sphere

$$i.e.\quad \frac{x^3}{125} = \frac{1372}{500}$$

$$\text{whence}\quad 4x^3 = 1372$$

$$\therefore\quad x^3 = 343$$

$$\therefore\quad x = 7 \text{ cms.}$$

107. Theorem.—If **P** and **Q** vary in such a way that their *ratio remains constant*, then **P** varies directly as **Q**.

PROOF.—Suppose that **P** and **Q** vary, but that their ratio, $\frac{\mathbf{P}}{\mathbf{Q}}$, is always equal to some fixed quantity m. Let q_1 * and q_2 be any two values of **Q**, and p_1 and p_2 the corresponding values of **P**.

Then we are told that $\frac{p_1}{q_1} = m$, and $\frac{p_2}{q_2} = m$.

$$\text{Since}\quad \frac{p_1}{q_1} = m; \quad \therefore p_1 = mq_1$$

$$\text{since}\quad \frac{p_2}{q_2} = m; \quad \therefore p_2 = mq_2$$

$$\text{hence}\quad \frac{p_1}{p_2} = \frac{mq_1}{mq_2}$$

$$i.e.\quad \frac{p_1}{p_2} = \frac{q_1}{q_2}$$

Thus the *ratio of any two values of* **P** *is equal to the ratio of the corresponding two values of* **Q**. Hence **P** varies directly as **Q**.

Conversely, if **P** varies directly as **Q**, then the ratio $\frac{\mathbf{P}}{\mathbf{Q}}$ remains constant.

* Note that q_1 and q_2 simply denote two different values of Q. The little figures $_1$ and $_2$ written below the letters are called "**subscripts.**" They *do not denote any operation*, but simply serve to distinguish q_1 from q_2; q_1 and q_2 may be treated like any other algebraical symbol; thus q_1^3 denotes $q_1 \times q_1 \times q_1$; q_1q_2 denotes $q_1 \times q_2$, etc.

If we represent this constant by the letter m, we have $\frac{\mathbf{P}}{\mathbf{Q}} = m$, which gives $\mathbf{P} = m\mathbf{Q}$.

If, then, we know that **P** and **Q** vary, and that m is constant, it follows that the equation $\mathbf{P} = m\mathbf{Q}$ is equivalent to the statement $\mathbf{P} \propto \mathbf{Q}$.

The examples worked out in § 106 may also be solved by using this principle. We will show the method by solving again Example (2).

EXAMPLE.—*The circumference of a circle varies directly as its diameter; and when the diameter is* 21 *inches the circumference is* 66 *inches: find the circumference when the diameter is* 35 *inches.*

Let **C** and **D** represent the circumference and diameter respectively.

$$\text{Then } \mathbf{C} \propto \mathbf{D}$$
$$\therefore \mathbf{C} = m\mathbf{D}, \text{ where } m \text{ is some constant}$$
$$\text{but when } \mathbf{D} \text{ is } 21,\ \mathbf{C} = 66$$
$$\therefore 66 = m \times 21$$
$$\text{hence } m = \tfrac{66}{21} = \tfrac{22}{7}$$
$$\therefore \mathbf{C} = \tfrac{22}{7}\mathbf{D}$$

[This equation gives the actual relation between the circumference and diameter of any circle.]

Thus when **D** is 35—

$$\mathbf{C} = \tfrac{22}{7} \times 35 = 110 \text{ inches}$$

108. We will work one or two more examples on direct variation by the latter method.

EXAMPLE (1).—*The volume of a sphere varies as the cube of its diameter; when the diameter is* 1 *inch, the volume is* ·5326 *cub. inch: find the volume when the diameter is* 2·5 *inches.*

Let **V** represent the volume and **D** the diameter, then $\mathbf{V} \propto \mathbf{D}^3$; thus—

$$\mathbf{V} = m\,\mathbf{D}^3, \text{ where } m \text{ is constant}$$
$$\text{but when } \mathbf{D} = 1,\ \mathbf{V} = {\cdot}5236$$
$$\therefore {\cdot}5236 = m \times 1$$
$$i.e.\ \ m = {\cdot}5236$$
$$\text{hence } \mathbf{V} = {\cdot}5236\,\mathbf{D}^3$$

[This equation gives the actual relation between the volume and diameter of a sphere.]

Thus when **D** is 2·5—

$$\mathbf{V} = {\cdot}5236 \times (2{\cdot}5)^3 = 8{\cdot}18125 \text{ cub. inches}$$

EXAMPLE (2).—*A light wire,* AB, *is stretched between two pegs,* A *and* B, *on the same horizontal level, and a weight is hung on to its middle point,* C. *Provided the weight is small, the downward displacement of* C *varies directly as the weight. When the weight is* 2·5 *gms., the downward displacement is* ·235 *cm.*

Find what weight will make the downward displacement equal to ·3 cm. (See Fig. 3.)

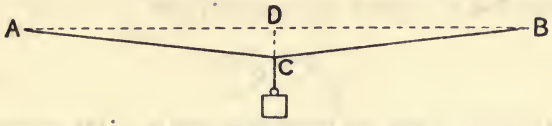

FIG. 3.

Let **D** be the middle point of the line **AB**. Then **DC** is the downward displacement of **C**. Thus, if **W** represent the weight, $DC \propto W$.

$$\text{Hence } DC = mW$$
$$\text{but when } W \text{ is } 2{\cdot}5,\ DC = {\cdot}235$$
$$\text{hence } {\cdot}235 = m \times 2{\cdot}5$$
$$\therefore m = {\cdot}235 \div 2{\cdot}5 = {\cdot}094$$
$$\text{thus we have } DC = {\cdot}094\ W$$

When the displacement is ·3—

$${\cdot}3 = {\cdot}094\ W$$
$$\therefore W = {\cdot}3 \div {\cdot}094 = {\cdot}31915 \text{ gms. nearly}$$

109. Inverse Variation—Definition.—One quantity is said to **vary inversely** as a second quantity if **the ratio of any two values of the first quantity is equal to the reciprocal* of the ratio of the corresponding values of the second quantity.**

Illustration.—If two men are running a mile, and if the ratio of their *rates* is 7 : 8, then the ratio of their *times* for the mile will be 8 : 7. (For the man whose rate is *less* will require *more* time.) Thus the time required to run a mile varies inversely as the rate of running.

EXAMPLE (1).—*If a garrison of* 1000 *men are provisioned for* 39 *days, how long will their provisions last if they are joined by another* 300 *men?*

The time which the food lasts varies inversely as the number of men.

Thus the time in the *second case* : the time in the *first case*
= the number of men in *first case* : the number of men in the *second case*

$$\text{i.e. } \frac{x}{39} = \frac{1000}{1300}$$
$$\therefore 100x = 3000$$
$$\therefore x = 30 \text{ days}$$

EXAMPLE (2).—*The pressure of a given quantity of gas varies inversely as its volume (provided the temperature is constant). A certain quantity of gas, when compressed into* 8 *cub. feet, exerts a pressure of* 30 *lbs. to the square inch: what pressure would it exert if compressed into* 5 *cub. feet?*

The pressure in the *second* case : the pressure in the *first* case
= the volume in the *first* case : the volume in the *second* case

$$\therefore \frac{x}{30} = \frac{8}{5}$$
$$\text{whence } x = 48 \text{ lbs. to the square inch}$$

---

* The reciprocal of the ratio $a : b$ is the ratio $b : a$.

110. Theorem.—**If P and Q vary in such a way that their product remains constant, then P varies inversely as Q.**

Suppose $\mathbf{PQ} = m$, where m is constant, then

$$\mathbf{P} = \frac{m}{\mathbf{Q}}$$

Thus if q_1, q_2 are any two values of **Q**, and p_1, p_2 the corresponding values of **P**, then—

$$p_1 = \frac{m}{q_1}$$

$$p_2 = \frac{m}{q_2}$$

$$\therefore \frac{p_1}{p_2} = \frac{m}{q_1} \div \frac{m}{q_2}$$

$$i.e. \quad \frac{p_1}{p_2} = \frac{q_2}{q_1}$$

Hence the ratio of any two values of **P** is the reciprocal of the ratio of the two corresponding values of **Q**; that is—

$\mathbf{P} \propto$ inversely as **Q**

Conversely, if $\mathbf{P} \propto$ inversely as **Q**, $\mathbf{PQ} = m$, where m is some constant, or—

$$\mathbf{P} = \frac{m}{\mathbf{Q}}$$

Theorem.—If $\mathbf{P} \propto$ inversely as Q, then $\mathbf{P} \propto$ directly as $\frac{1}{\mathbf{Q}}$.

For if $\mathbf{P} \propto$ inversely as **Q**—

then $\mathbf{PQ} = m$, where m is a constant
[by the preceding theorem.

$$\therefore \mathbf{P} = m \times \frac{1}{\mathbf{Q}}$$

i.e. $\mathbf{P} \propto$ directly as $\frac{1}{\mathbf{Q}}$
[by the theorem in § 107.

111. Note that if **P** and **Q** are known to vary, and m is known to remain constant, then the equation $\mathbf{P} = \frac{m}{\mathbf{Q}}$ is equivalent to the statement $\mathbf{P} \propto$ inversely as **Q**.

EXAMPLE (1).—*If* **K** *varies inversely as* $\mathbf{H}^2$, *and if* **K** *is* 6 *when* **H** *is* 6, *find* **K** *when* **H** *is* 4.

$\mathbf{K} = \frac{m}{\mathbf{H}^2}$, where m is a constant. But **K** is 6 when **H** is 6.

$$\therefore \ 6 = \frac{m}{36}$$

$$\text{hence } m = 216$$

$$\text{thus } \mathbf{K} = \frac{216}{\mathbf{H}^2}$$

[This equation expresses the actual relation between **K** and **H**.]
Thus, when $\mathbf{H} = 4$,

$$\mathbf{K} = \tfrac{216}{16} = \tfrac{27}{2}$$

EXAMPLE (2).—*The attractive force between two planets varies inversely as the square of the distance between them. If they attract each other with a force of* 1,000,000 *tons' weight when they are at a distance of* 400,000,000 *miles, with what force will they attract each other at a distance of* 50,000,000 *miles?*

Let **F** and **D** represent the force and distance respectively.
Then $\mathbf{F} \propto$ inversely as $\mathbf{D}^2$—

$$\text{hence } \mathbf{F} = \frac{m}{\mathbf{D}^2}$$

But when **D** is 400,000,000, **F** is 1,000,000. Thus—

$$1{,}000{,}000 = \frac{m}{(400{,}000{,}000)^2}$$

$$\therefore m = 1{,}000{,}000 \times (400{,}000{,}000)^2$$

$$\text{thus } \mathbf{F} = \frac{1{,}000{,}000 \times (400{,}000{,}000)^2}{\mathbf{D}^2}$$

Therefore, when **D** is 50,000,000—

$$\mathbf{F} = \frac{1{,}000{,}000 \times (400{,}000{,}000)^2}{(50{,}000{,}000)^2}$$

$$= 1{,}000{,}000 \times 8^2$$

$$= 64{,}000{,}000 \text{ tons' weight}$$

112. Collecting our formulæ—

If $\mathbf{P} \propto \mathbf{Q}$; then $\frac{p_1}{p_2} = \frac{q_1}{q_2}$; also $\mathbf{P} = m\mathbf{Q}$

If $\mathbf{P} \propto$ inversely as Q; then $\frac{p_1}{p_2} = \frac{q_2}{q_1}$; also $\mathbf{P} = \frac{m}{\mathbf{Q}}$

The student is strongly advised to learn *both methods* of working variation problems, as each method helps to make the theory more intelligible.

EXAMPLES.—LXXX.

1. If $\mathbf{P} \propto \mathbf{Q}$, and **P** is 15 when **Q** is 10, find **P** when **Q** is 36.
2. If $x \propto y$, and x is 10 when y is 8, find y when x is 45.
3. If $x \propto y$, and y is 12 when x is 16, find y when x is 36.
4. If $\mathbf{L} \propto \mathbf{M}$, and if **L** is ·246 when **M** is 3·69, find the relation between **L** and **M**, and find **M** when **L** is ·512.
5. If $\mathbf{P} \propto$ inversely as **Q**, and if **P** is 15 when **Q** is 10, find **P** when **Q** is 25.

6. If $y \propto$ inversely as z, and if z is 10 when y is 8, find z when y is 3·2.

7. If $x \propto$ inversely as y, and if y is 12 when x is 36, find y when x is 16.

8. If $L \propto$ inversely as M, and if L is ·246 when M is ·512, find the relation between L and M, and find M when L is ·369.

9. If $x \propto y^2$, and if x is 9 when y is 6, find x when y is 10.

10. If $P \propto Q^{\frac{1}{2}}$, and if P is 8 when Q is 10, find Q when P is 18.

11. If $L \propto$ inversely as M^3, and if L is 6 when M is 2, find the relation between L and M, and find the value of M when L is $\frac{16}{9}$.

12. If $L \propto$ inversely as $M^{\frac{3}{2}}$, and if L is 24 when M is 2·25, find L when M is 9.

13. If 10 cub. centimetres of gold weigh 193·6 gms., how many cubic centimetres will weigh 1 kilogram?

14. If 943 gallons of water flow past a water-mill in 23 minutes, how many gallons will flow past in 52 minutes?

15. If a train travels from London to Edinburgh in 8 hours 20 minutes, when its average rate is 50 miles an hour, in what time will it perform the journey if its average rate is 40 miles an hour?

16. 100 hurdles will just stretch across a field, when each hurdle measures 6 feet: how many hurdles will be required if each measures 5 feet?

17. The diagonal of a cube varies directly as the length of an edge. When the edge measures 3 inches, the diagonal measures 5·196 inches: find the diagonal when the edge measures 2·25 inches.

18. The surface of a sphere varies directly as the square of its diameter; if the diameter is 2 inches, the surface is 12·56637 sq. inches: find the surface when the diameter is 5 inches.

19. In the solar system, if T be the time of revolution of a planet in its orbit, and if D be the mean distance of that planet from the Sun, then $T \propto D^{\frac{3}{2}}$. Assuming that the Earth's period of revolution about the Sun is 365 days, find the period of Venus, given that the ratio of the mean distances of the Earth and Venus from the Sun is 25·18.

20. The pressure of a given quantity of gas at constant temperature varies inversely as its volume. A vessel contains 10 cub. feet of gas at a pressure of 300 lbs. per sq. inch: into what volume must the gas be compressed in order that its pressure may become 500 lbs. per sq. inch? What is its volume when the pressure is 120 lbs. per sq. inch?

21. The attractive force between two oppositely electrified balls varies inversely as the square of the distance between them. At a distance of 20 cms. the force is ·56 gms. weight: at what distance will the force be 2 gms. weight?

22. In order to make a series of circular copper coins, of the same weight, but of different shape, the thickness of the coins must vary inversely as the square of the diameter. If in one coin the thickness is ·05 inch and the diameter 1 inch, what will be the diameter of a coin whose thickness is ·08 inch? and what the thickness of a coin whose diameter is ·8 inch?

113. Compound Variation.—The value of one quantity often depends on the value of two or more other quantities. Suppose, for instance, that we are given the equation $P = m\frac{QR^{\frac{1}{2}}}{S^2}$, where m is a constant, and the other letters are variables, then P varies directly as the quantity $\frac{QR^{\frac{1}{2}}}{S^2}$. But this is often expressed by saying that—

P varies directly as Q,
directly as $R^{\frac{1}{2}}$, and
inversely as S^2.

Conversely, if we are told that V varies directly as X^2, inversely as Y, and inversely as $Z^{\frac{3}{2}}$, this statement means that $V \propto \frac{X^2}{YZ^{\frac{3}{2}}}$.

Using the principles of § 106, we may assume that—

$$v_1 : v_2 = \frac{x_1^2}{y_1 z_1^{\frac{3}{2}}} : \frac{x_2^2}{y_2 z_2^{\frac{3}{2}}}$$

$$\text{which gives } \frac{v_1}{v_2} = \frac{x_1^2}{x_2^2} \cdot \frac{y_2}{y_1} \cdot \frac{z_2^{\frac{3}{2}}}{z_1^{\frac{3}{2}}}$$

Or, using the principles of § 108, we may assume that $V = \frac{m\,X^3}{YZ^{\frac{3}{2}}}$ where m is some constant.

EXAMPLE (1).—*The volume of a drain-pipe varies directly as its length, and directly as the square of its diameter. When the length is* 20 *inches and the diameter* 5 *inches, the volume is* 392·7 *cub. inches. Find the volume when the length is* 12 *inches and the diameter* 6 *inches.*

Let V, L, D represent respectively the volume, length, and diameter. Then $V \propto LD^2$.

$$\therefore V = mLD^2$$

but when $L = 20$ and $D = 5$, then $V = 392{\cdot}7$

$$\text{hence } 392{\cdot}7 = m \times 20 \times 25$$
$$\text{whence } m = {\cdot}7854$$
$$\therefore V = {\cdot}7854 \times LD^2$$

Thus when L is 12 and D is 6—

$$V = {\cdot}7854 \times 12 \times 36 = 339{\cdot}3$$

EXAMPLE (2).—*The intensity of illumination varies inversely as the square of the distance of the source of light, and directly as the strength of the source of light. Two different sources of light are illuminating opposite sides of a screen; their strengths are* 15 *candle-power and* 30 *candle-power respectively; and their distances from the screen are in the ratio* 4 : 3. *Find the ratio of the intensities of illumination.*

Let I, D, S represent respectively the intensity of illumination, the distance of the source of light, and its strength. Then $I \propto \frac{S}{D^2}$. Thus—

$$i_1 : i_2 = \frac{s_1}{d_1^2} : \frac{s_2}{d_2^2}$$

$$\text{whence } \frac{i_1}{i_2} = \frac{s_1}{s_2} \cdot \frac{d_2^2}{d_1^2}$$

$$\text{but } \frac{s_1}{s_2} = \frac{15}{30} = \frac{1}{2}; \quad \frac{d_2}{d_1} = \frac{4}{3}$$

$$\text{hence } \frac{i_1}{i_2} = \frac{1}{2} \cdot \left(\frac{4}{3}\right)^2 = \frac{8}{9}$$

Thus the intensities are in the ratio 8 : 9.

EXAMPLES.—LXXXI.

1. If **P** varies directly as **Q** and inversely as **R**, and if **P** is 20 when **Q** is 10 and **R** is 2; find **P** when **Q** is 20 and **R** is 100.

2. If **P** varies directly as **Q**, directly as the square root of **R**, and inversely as the square root of **S**; and if **P** is 20 when **Q** is 10, **R** is 12, and **S** is 3; find **Q** when **P** is 5, **R** is 125, and **S** is 5.

3. If **V** varies inversely as **X**, directly as $\mathbf{Y}^2$, and inversely as $\mathbf{Z}^{\frac{1}{2}}$, and if **V** is 12 when **X** is 3, **Y** is 6, and **Z** is 9; find **V** when **X** is $\frac{1}{3}$, **Y** is 2, and **Z** is 16.

4. The weight of a copper coin varies directly as its thickness and as the square of its diameter. When the diameter is 3 cms. and the thickness ·2 cm., the weight is 11 gms.; find the weight when the diameter is 5 cms. and the thickness ·3 cm.

5. Using the data of the preceding question, find the thickness, if the diameter is 4 cms. and the weight 12 gms.

6. The resistance of the air on a spherical bullet varies directly as the square of its diameter and as the square of its velocity. If the resistance to a bullet, whose diameter is $\frac{1}{3}$ inch and whose velocity is 1000 feet per second, be 30 ozs. weight, find the resistance to a bullet whose diameter is 8 inches and whose velocity is 1200 feet per second.

7. From the data of the preceding question, find the diameter of a bullet which experiences a resistance of 60 ozs. weight when travelling at 800 feet per second.

8. The volume of a drain-pipe varies directly as the square of its diameter and as its length. When the length is 20 inches and the diameter 5 inches, the volume is 392·7 cub. inches. Find the length when the volume is 600 cub. inches and the diameter 6 inches.

9. The weight of a silver coin varies directly as its thickness and as the square of its diameter. If the thicknesses of two coins are in the ratio 3 : 5, and their diameters in the ratio 4 : 3, find the ratio of their weights.

10. The weight of a silver coin varies directly as its thickness and as the square of its diameter; if the weights of two coins are in the ratio 5 : 3, and their thicknesses are in the ratio 3 : 5, find the ratio of their diameters.

11. When a weight is suspended by an elastic string, the increase of length of the string varies directly as the weight and directly as the unstretched length of the string, and inversely as the square of the thickness of the string. If a weight of 2 lbs., attached to the end of a string whose unstretched length is 10 inches and whose thickness is ·1 inch, stretches it 2 inches; what will be the length of a string whose thickness is ·2 inch and whose unstretched length is 8 inches, when a weight of 5 lbs. is attached?

12. Two elastic strings of the same material support weights of 10 and 15 lbs. respectively; their unstretched lengths are 6 and 12 inches respectively, and their stretched lengths are 9 and 16 inches. If the thickness of the first string is ·3 inch, find that of the second string. (Use the variation statement in Question 11.)

13. Two elastic strings of the same material and of the same unstretched length support weights whose ratio is 3 : 4; if the thicknesses are in the ratio 4 : 3, and the increase in length of the first string is ·75 inch, find that of the second string.

14. Two elastic strings of the same material support equal weights. Their unstretched lengths are in the ratio 3 : 5, their thicknesses in the ratio 3 : 2; find the ratio of the increase of length of the first to the increase of length of the second.

113 (a). The relation between two quantities is often of a complex nature.

EXAMPLE.—*If two variable quantities, x and y, are connected by an equation of the form $y = px + qx^3$, where p and q are constant; find the values of p and q, given that when $x = 1$, $y = 5$, and when $x = 2$, $y = 28$.*

Substituting each set of values in the given equation, we obtain—

$$5 = p + q \quad \ldots \ldots \ldots \quad \text{(i.)}$$
$$28 = 2p + 8q \quad \ldots \ldots \ldots \quad \text{(ii.)}$$

Solving equations (i.) and (ii.) by the method of § 84, we find $p = 2$, $q = 3$. Thus the required equation is $y = 2x + 3x^2$.

EXAMPLES.—LXXXI. (*a*).

1. y and z are connected by an equation of the type $y = az + bz^2$. Find this equation, if $y = 6$ when $z = 1$, and $y = 2$ when $z = \cdot 5$.

2. **A** and **B** are connected by an equation of the type $\mathbf{A} = p\mathbf{B} + q\mathbf{B}^{-1}$. Find this equation, if $\mathbf{A} = \cdot 8$ when $\mathbf{B} = 1$, and $\mathbf{A} = 1\cdot 6$ when $\mathbf{B} = 3$.

3. **A** and **B** are connected by an equation of the type $\mathbf{A} = h + k\mathbf{B}^{\frac{3}{2}}$. Find this equation, if $\mathbf{A} = 2$ when $\mathbf{B} = 1$, and $\mathbf{A} = 12\cdot 5$ when $\mathbf{B} = 4$.

4. x and y are connected by an equation of the type $y = lx + m \,.\, 8^{-x}$. Find this equation, if $y = 2$ when $x = 0$, and $y = 2$ when $x = \frac{1}{3}$; and from the equation find y when $x = 1$.

5. The cost in shillings [**C**] of a polished cube of wood is connected with the length in feet of its edge [**L**] by an equation of the type $\mathbf{C} = a\mathbf{L}^2 + b\mathbf{L}^3$. When the edge is 2 feet, the cost is 48 shillings; when the edge is 5 feet, the cost is 675 shillings: what will be the cost when the edge is (i.) 1 foot, (ii.) 3 feet?

6. The value of **P** depends on the values of **Q** and **R**, and these quantities are connected by an equation of the type $P = aQ + bR^2$. Find this equation, if **P** is 5 when **Q** is 1 and **R** is ·5, and **C** is 2 when **Q** is ·5 and **R** is ·25.

7. If **H, K, L** are connected by an equation of the type $H = aK + bL^{-\frac{1}{2}}$; find this equation, given that **H** is 7 when **K** is 2 and **L** is 4, also **H** is 10 when **K** is 3 and **L** is 2·25.

8. If **X, Y, Z** are connected by an equation of the type $X = hYZ + kY^{-2}$; and if **X** is 3 when **Y** is 2 and **Z** is ·5, while **X** is 1·5 when **Y** is 4 and **Z** is ·25; find **Z** when **X** is 10 and **Y** is 1.

9. If the number of bricks [**N**] which a bricklayer can lay in a week is given by the formula $N = 3000\left\{1 - \frac{1}{64}(W-8)^2\right\}\left\{1 - \frac{1}{49}(S-7)^2\right\}$; where **W** denotes the number of hours' work which he does each day, and **S** the number of hours' sleep which he takes each night. Find the result, when (i.) $W = 6$, $S = 8$; (ii.) $W = 10$, $S = 6$; (iii.) $W = 8$, $S = 7$. Also show that, according to this formula, the values in (iii.) give the greatest possible result.

CHAPTER XIII.

ON LOGARITHMS.

114. Logarithms are indices used for purposes of rapid calculation.

Definition.—The logarithm of a number to a given base is the index of that power of the base which is equal to the number.

Thus the logarithm of 8 to base 2 is 3, because $2^3 = 8$, and therefore 3 *is the index of that power of* 2 *which is equal to* 8.

The logarithm of 8 to base 2 is usually expressed by the abbreviation $\log_2 8$.

EXAMPLE (1).—*Find* $\log_3 81$.

This is equivalent to the question, "What power of 3 is equal to 81?" By actual multiplication, we find that 81 is the fourth power of 3, *i.e.* $81 = 3^4$; hence $\log_3 81 = 4$.

EXAMPLE (2).—*Find* $\log_5 125$.

Since $125 = 5^3$; $\therefore \log_5 125 = 3$.

EXAMPLE (3).—*Find* $\log_8 4$.

Since 4 is the square of the cube root of 8; $\therefore 4 = 8^{\frac{2}{3}}$; $\therefore \log_8 4 = \frac{2}{3}$.

EXAMPLE (4).—*Find* $\log_3 \frac{1}{27}$.

$$27 = 3^3;\ \therefore \tfrac{1}{27} = 3^{-3} \text{ (see § 102)};\ \therefore \log_3 \tfrac{1}{27} = -3.$$

EXAMPLES.—LXXXII.

1. Find the logarithms to base 2 of the following quantities: 4, 16, 32, $\sqrt{2}$, $\sqrt[3]{2}$, $\frac{1}{2}$, ·25, ·125.

2. Find the logarithms to base 3 of the following quantities: 9, 243, 729, $\frac{1}{3}$, $\frac{1}{81}$, $\sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[5]{9}$.

3. Find the logarithms to base 4 of the following quantities: 16, 256, 2, 8, 128, ·25, ·0625, ·5, ·125.

4. Find the logarithms to base 9 of the following quantities: 729, $\frac{1}{9}$, $\frac{1}{81}$, 3, 27, $\frac{1}{3}$, $\frac{1}{729}$.

5. Evaluate $\log_{16} 2$, $\log_4 \cdot 5$, $\log_2 \cdot 25$, $\log_3 \dfrac{1}{\sqrt{3}}$.

6. Evaluate $\log_{10} 100$, $\log_{10} 1000$, $\log_{10} 1{,}000{,}000$, $\log_{10} \cdot 1$, $\log_{10} \cdot 01$, $\log_{10} \cdot 0001$.

115. The principles of the method by which these logarithms, or indices, may be used to facilitate calculation, can be explained by the use of the following table of logarithms to base 2 :—

Log.	Number.
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
11	2048
12	4096
13	8192
14	16384
15	32768
16	65536
17	131072
18	262144
19	524288
20	1048576

This table gives, for instance, that the logarithm of 8192 to base 2 is 13; and it must be remembered that this is equivalent to the statement $8192 = 2^{13}$.

EXAMPLE (1).—*Multiply* 256 *by* 1024.

From the table we see that the logarithm of 256 is 8; hence $256 = 2^8$; similarly, using the table, $1024 = 2^{10}$; thus

$$256 \times 1024 = 2^8 \times 2^{10}$$
$$= 2^{18} \qquad \text{[by First Index Law.}$$

But from the table, $2^{18} = 262144$, which is therefore the required product.

EXAMPLE (2).—*Evaluate* $\dfrac{128 \times 512 \times 32768}{1024 \times 262144 \times 2048}$.

Using our table, we find that this expression is equivalent to—

$$\frac{2^7 \times 2^9 \times 2^{15}}{2^{10} \times 2^{18} \times 2^{11}}$$
$$= 2^{7+9+15-10-18-11} \qquad \text{[by First Index Law.}$$
$$= 2^{-8} = \frac{1}{2^8}$$
$$= \tfrac{1}{256} \qquad \text{[using the table.}$$

EXAMPLE (3).—*Evaluate* $(64)^3$.

$$(64)^3 = (2^6)^3 \qquad \text{[using the table.}$$
$$= 2^{18} \qquad \text{[by Second Index Law.}$$
$$= 262144 \qquad \text{[using the table.}$$

EXAMPLE (4).—*Evaluate* $\sqrt[5]{1048576}$.

$$\sqrt[5]{1048576} = \sqrt[5]{2^{20}} \qquad \text{[using the table.}$$
$$= (2^{20})^{\frac{1}{5}}$$
$$= 2^4 \qquad \text{[by Second Index Law.}$$
$$= 16 \qquad \text{[using the table.}$$

EXAMPLE (5).—*Evaluate* $\dfrac{(256)^3 \times \sqrt[4]{65536}}{\sqrt[6]{262144} \times \sqrt[3]{32768}}$.

Using the table, this reduces to—

$$\frac{(2^8)^3 \times \sqrt[4]{2^{16}}}{\sqrt[6]{2^{18}} \times \sqrt[3]{2^{15}}} = \frac{(2^8)^3 \times (2^{16})^{\frac{1}{4}}}{(2^{18})^{\frac{1}{6}} \times (2^{15})^{\frac{1}{3}}}$$

$$= \frac{2^{24} \times 2^4}{2^3 \times 2^5} \quad \text{[by Second Index Law.}$$

$$= 2^{24+4-3-5} \quad \text{[by First Index Law.}$$

$$= 2^{20} = 1048576 \quad \text{[using the table.}$$

From these examples it should be obvious that this table enables us to perform very easily *any operations which include only multiplication, division, involution* (i.e. *squaring, cubing, etc.*), *and evolution* (i.e. *extraction of roots*); provided that all the quantities with which we are dealing occur in the list of numbers whose logarithms are given in the table.

EXAMPLES.—LXXXIII.

Calculate, by using the above table of logarithms to base 2 :—

1. 128×2048. **2.** 512×1024. **3.** 64×16384. **4.** $262144 \div 8192$.
5. $65536 \div 512$. **6.** $1048576 \div 16384$. **7.** $16384 \times 65536 \div 1024$.
8. $(1024)^2$. **9.** $(64)^3$. **10.** $(16)^4$. **11.** 8^6.
12. $\sqrt{65536}$. **13.** $\sqrt[3]{32768}$. **14.** $\sqrt[4]{1048576}$. **15.** $\sqrt[5]{32768}$.
16. $(32768)^{\frac{3}{5}}$. **17.** $(262144)^{\frac{5}{6}}$. **18.** $512 \times \sqrt{16384} \div \sqrt[3]{4096}$.

19. $\dfrac{8 \times \sqrt[3]{512} \times \sqrt[5]{32768}}{(128)^4}$. **20.** $\dfrac{(64)^2 \times \sqrt{4096} \times (1024)^3}{(16)^3 \times \sqrt[3]{262144} \times (512)^2}$.

116. The Laws of Logarithmic Calculation.—In using a table of logarithms, it is not usual to write out the work in the form of indices, as we have done in the last paragraph; but to convert the Index Laws into laws for logarithms (remembering that logarithms are indices).

From the First Index Law we derive the following

RULES.—**The logarithm of a product to any base is the sum of the logarithms of the factors to the same base.**

The logarithm of a quotient is found by subtracting the logarithm of the divisor from the logarithm of the dividend (all the logarithms being to the same base).

Expressing the rules as formulæ, we have—

$$\log_a (x \times y \times z) = \log_a x + \log_a y + \log_a z. \quad . \quad . \quad (1)$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, combining the rules—

$$\log_a \frac{x \times y}{p \times q} = \log_a x + \log_a y - \log_a p - \log_a q. \quad . \quad (3)$$

EXPLANATION.—By definition, $\log_a x$ is the index of that power of a which is equal to x. Thus—

$$x = a^{\log_a x}$$

$$\text{similarly, } y = a^{\log_a y}$$

$$z = a^{\log_a z}$$

$$\therefore x \times y \times z = a^{\log_a x} \times a^{\log_a y} \times a^{\log_a z}$$

$$i.e. \quad x \times y \times z = a^{\log_a x + \log_a y + \log_a z} \quad \text{[by First Index Law.}$$

Thus $\log_a x + \log_a y + \log_a z$ is the index of that power of a which is equal to $x \times y \times z$; *i.e.*

$$\log_a (x \times y \times z) = \log_a x + \log_a y + \log_a z$$

To prove formula (2).—As before—

$$x = a^{\log_a x}, \text{ and } y = a^{\log_a y} \qquad \text{[by Definition.}$$

$$\therefore \frac{x}{y} = \frac{a^{\log_a x}}{a^{\log_a y}}$$

$$i.e. \quad \frac{x}{y} = a^{\log_a x - \log_a y}$$

Thus $\log_a x - \log_a y$ is the index of that power of a which is equal to $\frac{x}{y}$;

$$i.e. \ \log_a \frac{x}{y} = \log_a x - \log_a y$$

From the Second Index Law we derive the

RULE.—**To find the logarithm of any power of a number, we multiply the logarithm of the number by the index of the power.**

Expressing the rule as a formula—

$$\log_a(x^k) = k \log_a x \quad . \ . \ . \ . \ . \ . \ . \ (4)$$

Proof.—As before—

$$x = a^{\log_a x} \qquad \text{[by Definition.}$$

$$\therefore x^k = (a^{\log_a x})^k$$

$$i.e. \quad x^k = a^{k \log_a x} \quad \text{[by Second Index Law.}$$

Thus $k \log_a x$ is the index of that power of a which is equal to x^k;

$$i.e. \quad \log_a (x^k) = k \log_a x$$

117. To show the use of these logarithm formulæ, let us apply them to the examples already worked out in § 115, still using the table of logarithms to base 2 which is given in that paragraph.

EXAMPLE (1).—*Multiply* 256 *by* 1024.

$$\log_2 (256 \times 1024) = \log_2 256 + \log_2 1024 \quad \text{[Formula (1).}$$
$$= 8 + 10 \quad \text{[from the table.}$$
$$i.e. \ \log_2 (256 \times 1024) = 18$$

Thus 256 × 1024 is the number whose logarithm to base 2 is 18, which (referring to the table) is 262144.

EXAMPLE (2).—*Evaluate* $\dfrac{128 \times 512 \times 32768}{1024 \times 262144 \times 2048}$.

$$\log_2 \frac{128 \times 512 \times 32768}{1024 \times 262144 \times 2048}$$
$$= \log_2 128 + \log_2 512 + \log_2 32768 - \log_2 1024 - \log_2 262144 - \log_2 2048$$ [by Formula (3).
$$= 7 + 9 + 15 - 10 - 18 - 11$$ [from the table.
$$= -8$$

Thus the logarithm of the required number to base 2 is -8; therefore, from the table, the required number is $\frac{1}{256}$.

[The table gives $\log_2 256 = 8$; hence $256 = 2^8$; $\therefore \frac{1}{256} = 2^{-8}$; $\therefore \log_2 \frac{1}{256} = -8$.]

EXAMPLE (3).—*Evaluate* $(64)^3$.

$$\log_2 (64)^3 = 3 \log_2 64$$ [by Formula (4).
$$= 3 \times 6$$ [from the table.
$$= 18$$

The logarithm of the required number is 18; therefore, from the table, the required number is 262144.

EXAMPLE (4).—*Evaluate* $\sqrt[5]{1048576}$.

$$\log_2 \sqrt[5]{1048576} = \log_2 (1048576)^{\frac{1}{5}}$$
$$= \tfrac{1}{5} \log_2 (1048576)$$ [by Formula (4).
$$= \tfrac{1}{5} \times 20$$ [from the table.
$$= 4$$

$\therefore$ the required number is 16 [from the table.

EXAMPLE (5).—*Evaluate* $\dfrac{(256)^3 \times \sqrt[4]{65536}}{\sqrt[6]{262144} \times \sqrt[3]{32768}}$.

The logarithm of this quantity to base 2—

$$= \log_2 (256)^3 + \log_2 (65536)^{\frac{1}{4}} - \log_2 (262144)^{\frac{1}{6}} - \log_2 (32768)^{\frac{1}{3}}$$ [by Formula (3).
$$= 3 \log_2 256 + \tfrac{1}{4} \log_2 65536 - \tfrac{1}{6} \log_2 262144 - \tfrac{1}{3} \log_2 32768$$ [by Formula (4).
$$= 3 \times 8 + \tfrac{1}{4} \times 16 - \tfrac{1}{6} \times 18 - \tfrac{1}{3} \times 15$$ [from the table.
$$= 24 + 4 - 3 - 5 = 20$$

Thus the required number is 1048576 [from the table.

If the student compares these with the solutions of the same problems in § 115, he will easily see that the method of working is really identical.

EXAMPLES.—LXXXIV.

Work again the questions in Examples LXXXIII., using the methods of § 117.

118. Common Logarithms.—The table in § 115 is obviously of no practical use, since we can only calculate by its aid provided that all the numbers involved in the calculation occur in the small list of numbers there given.

But advanced mathematical theory has enabled us to calculate the logarithm of any number to any base. The process is exceedingly laborious, and the explanation of it is beyond the scope of this book. But in Chambers's Mathematical Tables will be found a table of logarithms to base 10 of all numbers between 1 and 99,999. This table is all that is required for ordinary purposes of calculation.

For rough calculations it will be sufficient to use the table of logarithms correct to four figures, which is given at the end of this book.

Logarithms to base 10 are called common logarithms, and are by far the most useful for their purpose; where no base is mentioned, it is always to be understood that the base is 10; thus log 23 means $\log_{10} 23$.

Nearly all of these logarithms are fractions; and they are fractions which, like surds, can only be expressed as decimals correct to any required number of figures.

Thus, the table gives $\log_{10} 3 = \cdot 4771$, which means that $3 = 10^{\cdot 4771} = \sqrt[10000]{10^{4771}}$. This statement is, of course, not *perfectly* accurate, but it is so very approximate that it may be taken as accurate for most purposes.

We have already learnt the formulæ for logarithmic calculation in § 116. But it remains to learn the rules which enable us to obtain from the table the logarithm to base 10 of any quantity.

119. Integral Logarithms.—The following results depend on the principles of § 114; they should be committed to memory, as they are constantly required:—

$$\mathbf{1000 = 10^3;\ \therefore \log 1000 = 3.}$$
$$\mathbf{100 = 10^2;\ \therefore \log 100 = 2.}$$
$$\mathbf{10 = 10^1;\ \therefore \log 10 = 1.}$$
$$\mathbf{1 = 10^0;\ \therefore \log 1 = 0.}$$
$$\mathbf{\cdot 1 = \tfrac{1}{10} = 10^{-1};\ \therefore \log \cdot 1 = -1.}$$
$$\mathbf{\cdot 01 = \tfrac{1}{100} = 10^{-2};\ \therefore \log \cdot 01 = -2.}$$
$$\mathbf{\cdot 001 = \tfrac{1}{1000} = 10^{-3};\ \therefore \log \cdot 001 = -3.}$$

The list is obviously capable of being extended in both ways; thus $\log 10000000 = 7$; $\log \cdot 00001 = -5$, etc.

120. On Characteristics and Mantissæ.—The integral part of a logarithm is called its **characteristic**, and the decimal part is called its **mantissa**. For example—

$$\log 873{\cdot}05 = 2{\cdot}9410$$

Therefore the characteristic of log 873·05 is 2, and the mantissa of log 873·05 is ·9410. The table of logarithms only gives the *mantissa;* the *characteristic* can be supplied by a very easy rule.

If we are dealing with a negative logarithm, we always express it with a *positive mantissa* and a *negative characteristic* (the reason for this will appear later).

Thus log ·02 is $-1{\cdot}6989700$ (correct to seven decimal places); but we express this as $-2 + \cdot 3010300$, which obviously comes to the same thing; this is usually abbreviated $\mathbf{\bar{2}{\cdot}3010300}$, where, by placing the minus sign over the 2, we indicate that it refers to the 2 alone, and not to the decimal which follows it.

RULE.—**To express a negative logarithm as a logarithm with a positive mantissa, we add and subtract the next highest integer.** **(5)**

EXAMPLE (1). $-4{\cdot}5213487.$

$$= -5 + 5 - 4{\cdot}5213487 \qquad (\text{for } -5+5=0)$$
$$= -5 + {\cdot}4786513 = \bar{5}{\cdot}4786513$$

It is often necessary to add or subtract such logarithms, or to multiply or divide them by integers. The first three processes require no special rules.

EXAMPLE (2).—*From the sum of* $\bar{5}{\cdot}2303052$ *and* $2{\cdot}3048121$ *subtract* $\bar{1}{\cdot}2341112$.

This gives $(-5 + {\cdot}2303052) + (2{\cdot}3048121) - (-1 + {\cdot}2341112)$.

$$= -5 + {\cdot}2303052 + 2{\cdot}3048121 + 1 - {\cdot}2341112$$
$$= -4 + 2{\cdot}3010061 = -2 + {\cdot}3010061 = \bar{2}{\cdot}3010061$$

EXAMPLE (3).—*Multiply* $\bar{2}{\cdot}4771213$ *by* 4.

$$(-2 + {\cdot}4771213) \times 4 = -8 + 1{\cdot}9084852$$
$$= \bar{7}{\cdot}9084852$$

The method of dividing such a logarithm by an integer must be carefully noted; it is most easily explained by an example.

EXAMPLE (4).—*Divide* $\bar{4}{\cdot}3421815$ *by* 3.

$$(-4 + {\cdot}3421815) \div 3 = (-6 + 2{\cdot}3421815) \div 3$$
$$= -2 + {\cdot}7807272 = \bar{2}{\cdot}7807272$$

Note that -4 does not divide exactly by 3; the next highest negative integer which does divide exactly by 3 is -6; hence we replace -4 by $-6 + 2$.

EXAMPLES.—LXXXV.

1. Add together $\bar{2}{\cdot}3010300$, $3{\cdot}4771213$, $\bar{4}{\cdot}6020600$.

2. From the sum of $\bar{2}{\cdot}3010300$ and $3{\cdot}4771213$ subtract the sum of $\bar{2}{\cdot}6020600$ and $4{\cdot}6989700$.

3. Multiply $\bar{3}{\cdot}7212408$ by 3; and $\bar{1}{\cdot}9230705$ by 4.

4. Divide $\bar{3}{\cdot}4283071$ by 2; $\bar{5}{\cdot}2030418$ by 4; $\bar{3}{\cdot}2040713$ by 10.

5. Subtract $\bar{3}{\cdot}4771213$ from $\bar{4}{\cdot}3010300$.

6. Subtract $\bar{3}{\cdot}4771213$ from $\bar{1}{\cdot}3010300$.

121. Theorem.—If two numbers contain the same significant figures in the same order, but differ in the position of the decimal point, their logarithms to base 10 differ by an integer.

EXPLANATION.—Suppose the two numbers are $2362{\cdot}3$ and $2{\cdot}3623$; then—

$$2362{\cdot}3 = 2{\cdot}3623 \times 1000 \qquad [\S\ 4.$$
$$\therefore \log 2362{\cdot}3 = \log 2{\cdot}3623 + \log 1000 \qquad [\text{Formula } (1).$$
$$i.e. \quad \log 2362{\cdot}3 = \log 2{\cdot}3623 + 3 \qquad [\S\ 119.$$

Hence the logarithms of these two numbers differ by the integer 3.

It follows that two such numbers will have the same mantissa. For two quantities involving decimals, which differ by an integer, must obviously have the same figures after the decimal point.

This property of logarithms to base 10 is very important; it does not apply to logarithms to any other base; and it is for this reason that logarithms to base 10 are more convenient than logarithms to any other base. Moreover, in connection with this theorem, we can explain **the reason for representing a negative logarithm to base 10 with a positive characteristic** (see § 120). Let us find log ·0023623, given that log 2362·3 = 3·3733350.

$$\cdot0023623 = \frac{2362\cdot3}{1000000} \qquad [\S\ 4.$$

$$\therefore \log \cdot0023623 = \log 2362\cdot3 - \log 1000000 \quad [\text{Formula } (2).$$
$$= 3\cdot3733350 - 6 \qquad [\text{see } \S\ 119.$$
$$= +3 - 6 + \cdot3733350$$
$$= -3 + \cdot3733350 = \bar{3}\cdot3733350$$

Thus when we express the logarithm of ·0023623 with a positive mantissa, that mantissa is the same as the mantissa of log 2362·3; and in general *the logarithm to base* 10 *of any number whose significant figures are* 23623 *will have the same mantissa* ·3733350, *provided that it is always expressed with a positive mantissa.*

This mantissa will be found in the tables opposite the number 23623; and the characteristic of any number which has the same significant figures can be determined by the rules of the following paragraph.

122. Determination of Characteristics.

RULE.—**If a number is greater than unity, the characteristic of its logarithm to base 10 is positive, and is one less than the number of integers before the decimal point.** **(6)**

EXAMPLES.—In $\log_{10}$ 3523·68, the number of integers before the decimal point is four; hence the characteristic is 3. In log 3·52368 the number of integers before the decimal point is one; hence the characteristic is 0.

EXPLANATION.—3523·68 lies between 1000 and 10,000.
∴ its logarithm lies between log 1000 and log 10000
i.e. its logarithm lies between 3 and 4 [§ 119.
hence the characteristic is 3.
Similarly, log 3·52368 lies between log 1 and 10
i.e. between 0 and 1 [§ 119.
∴ the characteristic is 0.

RULE.—**If a number is less than unity, the characteristic of its logarithm to base 10 is negative, and is one more than the number of cyphers between the decimal point and the first significant figure.** **(7)**

EXAMPLES.—In log ·003428 the number of cyphers between the decimal point and the first significant figure is two; hence the characteristic is $\bar{3}$.

In log ·30428 the number of cyphers between the decimal point and the first significant figure is 0; hence the characteristic is $\bar{1}$.

EXPLANATION.—·003428 lies between ·001 and ·01.

∴ log ·003428 lies between log ·001 and log ·01
i.e. between −3 and −2. [§ 119.

Thus, *since the mantissa is to be kept positive*, the characteristic is −3.
[*e.g.* $-3 + \cdot 25 = -2\cdot 75$, which lies between −2 and −3.]
Similarly, log ·30428 lies between log ·1 and log 1
i.e. between −1 and 0 [§ 119.
∴ the characteristic is −1.

Note that the two preceding rules can be expressed in a tabular form which is easily remembered—

If the first significant figure of the number is in the	the characteristic of the logarithm is
thousands' place	3
hundreds' place	2
tens' place	1
units' place	0
1st decimal place	$\bar{1}$
2nd decimal place	$\bar{2}$
3rd decimal place	$\bar{3}$
and so on	

123. On the Table of Four-figure Logarithms.—For purposes of comparatively rough calculations, it is sufficient to use logarithms correct to four figures only. The table of four-figure logarithms is given at the end of this book.*

We have already mentioned that the table registers the mantissa only, and that the mantissa depends on the significant figures of the number, and not on the position of the decimal point in the number (§ 121).

The first two significant figures of the number are given in the first column; the third significant figure is given in the first portion of the top row, and the fourth significant figure is given in the latter portion of the top row.

The mantissa of a number of three significant figures is in the row

* For the Examination of the Science and Art Department on Elementary Practical Mathematics, it will be sufficient for a student to use this table only, without troubling about the ordinary seven-figure logarithms.

corresponding to the first two figures, and in the column corresponding to the third.

EXAMPLES.—(1) *To find log* 32500.

The significant figures are 325. Find 32 in the first column, and 5 in the first portion of the top row; the mantissa which lies in the same row as the 32, and in the same column as the 5, is ·5119. This is the mantissa corresponding to the figures 325.

By Rule (6), § 122, the characteristic of log 32500 is 4.

Hence log 32500 = 4·5119.

(2) *Find log* ·00037.

The significant figures are 37, which we count as 370; in the row marked 37 and in the column marked 0, we find the mantissa ·5682.

By Rule (7), § 122, the characteristic of log ·00037 is $\bar{4}$.

Hence log ·00037 = $\bar{4}$·5682.

The last nine columns in the table give the **figures which are to be added** to the mantissa of a number containing **three** significant figures to obtain the correct mantissa for a **fourth** significant figure.

EXAMPLES.—(3) *Find log* 530700.

The significant figures are 5307.

The mantissa for 530 is found in the row marked 53 and in the column marked 0, and is ·7243.

To this mantissa we add the figure found in the row marked 53, and in the *second column which is marked* 7; this figure is 6, and really counts as ·0006. Thus—

$$\begin{array}{r} \cdot 7243 \\ 6 \\ \hline \cdot 7249 \end{array} = \text{mantissa for } 5307$$

Thus, using Rule (6), log 530700 = 5·7249.

(4) *Find log* ·01358.

The mantissa for 135 is found in the row marked 13 and in the *first* column marked 5; this is ·1303.

To this mantissa we add the figures found in the row marked 13 and in the *second* column marked 8; these figures are 26, which count as ·0026. Thus—

$$\begin{array}{r} \cdot 1303 \\ 26 \\ \hline \cdot 1329 \end{array} = \text{mantissa for } 1358$$

Hence, using Rule (7), log ·01358 = $\bar{2}$.1329.

EXAMPLES.—LXXXVI.

Use the tables at the end of the book to find the logarithms of the following numbers:—

1. 657.	**2.** 283.	**3.** 52·8.	**4.** ·712.
5. ·0245.	**6.** 31200.	**7.** 6284.	**8.** ·02378.
9. ·3629.	**10.** 939200.	**11.** ·008382.	**12.** ·8005.
13. ·00032.	**14.** 333000.	**15.** 7777.	**16.** ·000000123.
17. 135200.	**18.** 63·25.	**19.** 303500.	**20.** ·0027.
21. $3{\cdot}25 \times 10^{10}$.	**22.** $4{\cdot}682 \times 10^{-11}$.	**23.** $7{\cdot}5 \times 10^{-6}$.	**24.** $6{\cdot}66 \times 10^{8}$.

124. The Table of Anti-logarithms.—Immediately after the Table of Logarithms at the end of this book will be found the Table of Anti-logarithms. We use this to find the number corresponding to a given logarithm. The construction of the table is similar to that of the Table of Logarithms.

EXAMPLES.—(1) *Find the number whose logarithm is* 2·593.

The significant figures which correspond to the mantissa ·593 are found in the row which is marked ·59, and in the first column which is marked 3; these figures are 3917.

Since the characteristic of the logarithm is 2, the required number has three figures before the decimal point, by Rule (6).

Thus the required number is 391·7.

Note particularly that the mantissa determines the significant figures of the required number, and the characteristic determines the position of the decimal point.

(2) *Find the number whose logarithm is* $\bar{3}$·8458.

To find the significant figures corresponding to the mantissa ·8458, we take the significant figures in the row which is marked ·84, and in the *first column* which is marked 5; these figures are 6998. To these significant figures we add the quantity found in the row which is marked ·84 and in the *second column* which is marked 8; viz. 13. Thus—

$$\begin{array}{r} 6998 \\ 13 \\ \hline 7011 \end{array}$$

7011 = significant figures which correspond to the mantissa ·8458.

Since the characteristic is $\bar{3}$, the required number must have two cyphers after the decimal point, by Rule (7).

Thus the required number = ·007011.

EXAMPLES.—LXXXVII.

Given the following logarithms, find from the table their anti-logarithms (*i.e.* the numbers of which these are the logarithms):—

1. 1·283.	**2.** 3·141.	**3.** ·087.	**4.** $\bar{1}$·939.	**5.** ·859.
6. $\bar{2}$·147.	**7.** ·3456.	**8.** 3·0487.	**9.** $\bar{2}$·9008.	**10.** 2·1122.
11. $\bar{5}$·8339.	**12.** $\bar{1}$·7236.	**13.** $\bar{3}$·024.	**14.** 2·2.	**15.** $\bar{3}$·7.
16. $\bar{5}$.	**17.** 10·3834.	**18.** $\bar{8}$·2371.	**19.** 11·9395.	**20.** $\overline{12}$·2438.
21. 7·77.	**22.** $\bar{8}$·888.	**23.** $\bar{9}$·259.	**24.** 20·3333.	

125. We can now work any calculation to four significant figures.—The fourth significant figure may not be correct, but will not be far wrong, unless the calculation is exceedingly complicated.

EXAMPLE (1).—*Evaluate* $\dfrac{34 \times 127 \times 378{\cdot}3}{2345 \times 1{\cdot}02}$.

$$\log \frac{34 \times 127 \times 378{\cdot}3}{2345 \times 1{\cdot}02}$$
$= \log 34 + \log 127 + \log 378{\cdot}3 - \log 2345 - \log 1{\cdot}02$ [Formula (3).
$= 1{\cdot}5315 + 2{\cdot}1038 + 2{\cdot}5778 - 3{\cdot}3701 - {\cdot}0086$ [from the table.
$= 2{\cdot}8344$

Thus 2·8344 is the logarithm of the required result. We therefore use the Table of Anti-logarithms to determine the number whose logarithm is 2·8344. By the method of § 124, this gives the result 682·9.

EXAMPLE (2).—*Evaluate* $(2{\cdot}05)^{20}$.

$$\log (2{\cdot}05)^{20} = 20 \log 2{\cdot}05 \qquad \text{[by Formula (4).}$$
$$= 20 \times {\cdot}3118 \qquad \text{[from the tables.}$$
$$= 6{\cdot}236$$

The anti-logarithm of 6·236 is 1722000 [from the table, and Rule (6).

EXAMPLE (3).—*Evaluate* $\sqrt[17]{100}$.

$$\log (100)^{\frac{1}{17}} = \tfrac{1}{17} \log 100 \qquad \text{[by Formula (4).}$$
$$= \tfrac{1}{17} \times 2 \qquad \text{[§ 119.}$$
$$= {\cdot}1176 \text{ correct to four figures}$$

The anti-logarithm of ·1176 is 1·311 [from the table, and Rule (6).

EXAMPLE (4).—*Evaluate* $\sqrt[3]{{\cdot}00003241}$.

$$\log \sqrt[3]{{\cdot}00003241} = \tfrac{1}{3} \log {\cdot}00003241 \qquad \text{[by Formula (4).}$$
$$= \tfrac{1}{3} \times \bar{5}{\cdot}5106 = \bar{2}{\cdot}5035$$

The anti-logarithm of $\bar{2}{\cdot}5035$ is ·03188 [from the table, and Rule (7).

EXAMPLE (5).—*Evaluate* $\dfrac{(20{\cdot}34)^{1{\cdot}32}}{(22{\cdot}58)^{4{\cdot}2}}$.

$$\log \frac{(20{\cdot}34)^{1{\cdot}32}}{(22{\cdot}58)^{4{\cdot}2}} = \log (20{\cdot}34)^{1{\cdot}32} - \log (22{\cdot}58)^{4{\cdot}2} \qquad \text{[by Formula (2).}$$
$$= 1{\cdot}32 \times \log (20{\cdot}34) - 4{\cdot}2 \times \log (22{\cdot}58) \qquad \text{[by Formula (4).}$$
$$= 1{\cdot}32 \times 1{\cdot}3083 - 4{\cdot}2 \times 1{\cdot}3537$$
$$= -3{\cdot}9586 \qquad \text{correct to four figures}$$
$$= \bar{4}{\cdot}0414$$

The anti-logarithm of $\bar{4}{\cdot}0414$ is ·00011 [from the table, and Rule (7).

EXAMPLES.—LXXXVIII.

Evaluate the following quantities by means of the four-figure logarithm tables :—

1. $2{\cdot}135 \times 5{\cdot}823$.

2. $5{\cdot}121 \times 1{\cdot}036$.

3. $16420 \times 16{\cdot}49$.

4. ${\cdot}002612 \times {\cdot}0008125$.

5. $22{\cdot}34 \div 5{\cdot}023$.

6. ${\cdot}3457 \div {\cdot}02346$.

7. $22000 \times 44090 \div 2174$.

8. ${\cdot}3202 \times {\cdot}02113 \div {\cdot}4114$.

9. ${\cdot}1613 \times 2384 \div {\cdot}02317$.

10. $200{\cdot}2 \div 1014 \times {\cdot}02159$.

11. $(2519)^2$. **12.** $({\cdot}7213)^2$.

13. $(1{\cdot}014)^{50}$. **14.** $\sqrt[3]{21{\cdot}87}$.

15. $\sqrt[20]{1587000000}$.

16. $\sqrt[20]{{\cdot}00001383}$.

17. $(2124)^2 \times (3{\cdot}045)^{\frac{2}{3}} \div (22{\cdot}15)^3$.

18. $(2{\cdot}135)^3 \times ({\cdot}00001258)^{\frac{1}{3}} \div (22{\cdot}82)^2$.

19. $\dfrac{\sqrt[3]{352} \times \sqrt[5]{32500}}{(13{\cdot}2)^4}$.

20. $\dfrac{(31)^2 \times \sqrt{4004} \times (2012)^3}{(17)^3 \times \sqrt[5]{28310} \times (666)^2}$.

21. $(1{\cdot}358)^{1{\cdot}23}$.

22. $(3{\cdot}297)^{-3{\cdot}15}$.

23. $(2{\cdot}483)^{1{\cdot}5} \div (3{\cdot}216)^{-2{\cdot}35}$.

24. $3^{1{\cdot}56} \times 7^{2{\cdot}4} \div 5^{1{\cdot}2}$.

126. On Using the Table of Seven-figure Logarithms.

No.	0	1	2	3	4	5	6	7	8	9	D.	
3431	5354207	4334	4460	4587	4713	4840	4967	5093	5220	5346		
32	5473	5599	5726	5852	5979	6105	6232	6359	6485	6612		
33	6738	6865	6991	7118	7244	7371	7497	7623	7750	7876	126	
34	8003	8129	8256	8382	8509	8635	8762	8888	9015	9141	1	13
35	9267	9394	9520	9647	9773	9900	$\overline{0026}$	$\overline{0152}$	$\overline{0279}$	$\overline{0405}$	2	25
36	5360532	0658	0784	0911	1037	1163	1290	1416	1543	1669	3	38
37	1795	1922	2048	2174	2301	2427	2553	2680	2806	2932	4	50
38	3059	3185	3311	3438	3564	3690	3817	3943	4069	4195	5	63
39	4322	4448	4574	4701	4827	4953	5079	5206	5332	5458	6	76
3440	5584	5711	5837	5963	6089	6216	6342	6468	6594	6721	7	88
41	6847	6973	7099	7225	7352	7478	7604	7730	7856	7982	8	101
42	8109	8235	8361	8487	8613	8739	8866	8992	9118	9244	9	113
43	9370	9496	9622	9749	9875	$\overline{0001}$	$\overline{0127}$	$\overline{0253}$	$\overline{0379}$	$\overline{0505}$		
44	5370631	0758	0884	1010	1136	1262	1338	1514	1640	1766		

The above is an extract from the Table of Logarithms.

The first four significant figures in the number are given in the first column, the fifth significant figure is given in the top line; the mantissæ are given in the second and following columns; the last column will be explained later.

The first three figures of the mantissæ are the same for many successive numbers, and are therefore not repeated.

Thus, if we want the mantissa of log 34400, we look for it in the line corresponding to the figures 3440, and in the column corresponding to the figure 0; this gives us ·5365584 (the 536 which is given four lines higher up belongs to all the mantissæ in this neighbourhood).

Similarly, the mantissa of log 34423 is found in the line corresponding to 3442 and the column corresponding to 3, and is therefore ·5368487.

The mantissæ of log 34434 and log 34435 are respectively ·5369875, and ·5370001, the line over the 0001 being used to denote that we are to take the first three figures of the mantissa from the line below.

EXAMPLE.—*To find log* ·00034438.

By Rule (7) the characteristic is −4.

The mantissa is the same as that of log 34438, which is given above as ·5370379.

Hence log ·00034438 = $\bar{4}$·5370379.

127. The reverse process is equally important.—*Notice that there is no table of anti-logarithms.*

EXAMPLE (1).—*Find the number whose logarithm is* 3·5359900.

From the above extract we find that the mantissa ·5359900 corresponds to the figures 34355.

Again, since the characteristic is 3, the number contains four figures before the decimal point.

Thus 3·5359900 = log 3435·5.

EXAMPLE (2).—*Find the number whose logarithm is* 3·5362838.

The mantissa ·5362838 does not occur in the table. The nearest to it which does occur is ·5362806, and this corresponds to the numbers 34378. By Rule (7) we insert two cyphers after the decimal point.

Thus the number whose logarithm is $\bar{3}\cdot5362838$ is ·0034378 *correct to five significant figures.*

EXAMPLES.—LXXXIX.

Find from a table of logarithms the values of—

1. log 3·8214. **2.** log 232·15. **3.** log 58140. **4.** log 32875000.
5. log 218700. **6.** log ·31416. **7.** log ·0034597. **8.** log ·000045.
9. log 32·586. **10.** log 2·8001. **11.** log ·0345. **12.** log 21435.

Find from a table of logarithms the numbers whose logarithms are—

13. 2·4476386. **14.** 4·4885789. **15.** $\bar{2}\cdot5210858$.
16. $\bar{4}\cdot5440308$. **17.** 6·7190411. **18.** ·5427259.

Find, correct to five significant figures, the numbers whose logarithms are—

19. 2·2871476. **20.** $\bar{2}\cdot1481715$. **21.** $\bar{4}\cdot2148975$.
22. ·3456789. **23.** 6·4120315. **24.** 5·3014201.

128. We can now work any calculation correct to five significant figures.

We use, as before, the formulæ of § 116.

EXAMPLE (1).—*Evaluate* $\dfrac{34 \times 127 \times 378\cdot32}{2345 \times 1\cdot0204}$.

$$\log \frac{34 \times 127 \times 378\cdot32}{2345 \times 1\cdot0204}$$

$= \log 34 + \log 127 + \log 378\cdot32 - \log 2345 - \log 1\cdot0204$ [by Formula (3).
$= 1\cdot5314789 + 2\cdot1038037 + 2\cdot5778593 - 3\cdot3701428 - \cdot0087704$
[from the table.
$= 2\cdot8342287$

Finding, from the tables, the number whose logarithm is 2·8342287, we obtain 682·70 correct to five significant figures.

EXAMPLE (2).—*Evaluate* $\dfrac{(20\cdot34)^{1\cdot32}}{(22\cdot58)^{4\cdot2}}$

$\log \dfrac{(20\cdot34)^{1\cdot32}}{(22\cdot58)^{4\cdot2}} = \log (20\cdot34)^{1\cdot32} - \log (22\cdot58)^{4\cdot2}$ [by Formula (2).

$= 1\cdot32 \times \log (20\cdot34) - 4\cdot2 \times \log (22\cdot58)$ [by Formula (4).
$= 1\cdot32 \times 1\cdot3083509 - 4\cdot2 \times 1\cdot3537239$
$= -3\cdot9586172 = \bar{4}\cdot0413828$

Which is the logarithm of ·00011000 correct to five significant figures.

EXAMPLES.—XC.

Evaluate the following quantities correct to five significant figures:—

1. $2\cdot1352 \times 5\cdot8235$. **2.** $16421 \times 16\cdot495$.
3. $22\cdot341 \div 5\cdot0231$. **4.** $22003 \times 44098 \div 21736$.
5. $\cdot16125 \times 2384\cdot2 \div \cdot023172$. **6.** $(25187)^2$.
7. $\sqrt[3]{21\cdot875}$. **8.** $\sqrt[20]{\cdot000013825}$.
9. $(2\cdot135)^3 \times (\cdot00001258)^{\frac{1}{3}} \div (22\cdot825)^2$.
10. $\dfrac{(31)^2 \times \sqrt{4004} \times (2012)^3}{(17)^3 \times \sqrt[5]{28315} \times (666)^2}$.
11. $(3\cdot2974)^{-3\cdot15}$. **12.** $3^{1\cdot56} \times 7^{2\cdot4} - 5^{1\cdot2}$.

129. On the Logarithms of Numbers containing Six Significant Figures.—If we turn again to the extract from the tables given in § 126, we shall find that the last column (marked D) will enable us to find the logarithm of a number of six significant figures.

The figure 126 at the top of this column merely denotes that consecutive mantissæ on this part of the page differ by 126, or more correctly by ·0000126. Below this number are given the quantities which must be added to the mantissa of a number of five significant figures to find the correct mantissa for a sixth significant figure.

EXAMPLE (1).—*Find log* 34·3368.

The mantissa for the figures 34336 is ·5357497; opposite the figure **8** in the last column we find the figures 101,

$$\begin{array}{r} \cdot 5357497 \\ 101 \\ \hline \cdot 5357598 \end{array} = \text{mantissa for } 343368$$

Hence log 34·3368 = 1·5357598.

EXAMPLE (2).—*Find, correct to six significant figures, the number whose logarithm is* $\bar{2}\cdot3481215$.

The nearest mantissa below ·3481215 which is given in the extract is ·3481101, which is the mantissa of 22290.

·3481215 exceeds ·3481101 by (·0000)**114**.

From the last column we see that an increase of **117** in the mantissa gives the sixth significant figure to be **6**.

Thus ·3481215 is the mantissa of 22290**6**, correct to six significant figures.

$$\therefore \bar{2}\cdot3481215 = \log \cdot0222906$$

EXAMPLES.—XCI.

Evaluate correct to six significant figures—

1. $32\cdot248 \times \cdot273412$.
2. $32\cdot248 \div \cdot273412$.
3. $263\cdot216 \times (\cdot2345)^2$.
4. $281\cdot314 \div (\cdot2345)^2$.
5. $\sqrt[7]{2385\cdot46}$
6. $\sqrt[20]{\cdot000415873}$.
7. $(6\cdot83457)^{\frac{10}{13}}$.
8. $(\cdot345679)^{-\frac{3}{5}}$.
9. $(\cdot345679)^{\frac{5}{3}}$.
10. $(231257000000)^{\frac{1}{12}}$.

130. Note on Hyperbolic Logarithms.—The only easy method of calculating logarithms accurately which has been discovered does not give the logarithms to the base 10, but to a base which is represented by the infinite series $1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \text{etc.}$ If this series is worked out correctly to seven significant figures, it gives 2·718282. The above series is of great importance in higher mathematical theory, and is always represented by the letter *e*.

When we have the table of logarithms to base *e*, we can obtain those to base 10 by dividing each logarithm to base *e* by $\log_e 10$, *i.e.* by multiplying each logarithm to base *e* by $\frac{1}{\log_e 10}$, which is ·434294482. The proof of this statement will be found in any mathematical treatise on the theory of logarithms.

Logarithms to base *e* are called hyperbolic logarithms, or sometimes Napierian logarithms, in honour of the inventor of logarithms. They are never used for calculating purposes, because they are comparatively inconvenient. For the theorems of §§ 121, 122, only hold for logarithms to base 10, and it is obvious that without these theorems our method of work would have to be seriously modified.

Many mathematical formulæ involving logarithms are, however, more conveniently expressed with logarithms to base *e*.

It is therefore useful for the student to remember that he can obtain the **Napierian logarithm of any number by multiplying the common logarithm of that number by 2·3026.**

Thus $\log_e 3 = \log 3 \times 2{\cdot}3026 = {\cdot}4771213 \times 2{\cdot}3026$.

$\log_e 300 = \log 300 \times 2{\cdot}3026 = 2{\cdot}4771213 \times 2{\cdot}3026$.

CHAPTER XIV.

THE SLIDE RULE.

131. Description of the Instrument.—The slide rule is an instrument by which a large number of calculations can be performed mechanically. The results are usually correct to three significant figures, which is accurate enough for ordinary practical work.

It consists of a ruler, **HKLM** (see Fig. 4), containing a groove, **NTVX**, in which runs a slide of the same length as the ruler. On the instrument are four graduated scales, **AA BB, CC, DD**, in the position shown in the diagram; viz. **AA, DD** on the ruler, one on each side of the groove, and **BB, CC** on the slide. Fig. 5 represents the instrument with the slide **PQRS** projecting to the right from the groove. **AA** and **BB** are graduated with numbers from 1 to 100; **CC** and **DD** are graduated with numbers from 1 to 10. When the slide does not project from the ruler, as in Fig. 4, **the graduations on BB coincide exactly with those on AA, and the graduations on CC coincide exactly with those on DD.**

The essential feature of the slide rule is the manner in which these scales are graduated. Note carefully that the graduations do not run from 0 to 100, or from 0 to 10; but from 1 to 100, and from 1 to 10.

Fig. 6 represents the scale **CC**; **E** is the point marked "1," and **F** is the point marked "10"; the scale is graduated in such a way that the *distance of any number from* **E** *is proportional to the logarithm of that number.* Now, the values of—

	log 1,	log 2,	log 3,	log 4,	log 5,	log 6,	log 7,	log 8,	log 9,	log 10,
are respectively	0,	·301,	·477,	·602,	·699,	·778,	·845,	·903,	·954,	1,
which are proportional to	0,	301,	477,	602,	699,	778,	845,	903,	954,	1000.

We therefore divide **EF** into 1000 equal parts, and then put the graduation "1" at **E**, the graduation "2" at the 301st division, the graduation "3" at the 477th division, etc.; the intermediate graduations are set off on the same principle; *e.g.* since log 3·12 = ·494, the graduation 3·12 is placed at the 494th division. As many graduations as possible are marked off in this manner.

It follows that the distance of any graduation from **E** *represents on a proportional scale the logarithm of the number at that graduation.*

Note that as the distance between the graduations "1" and "2" is much greater than that between the graduations "7" and "8," many more subdivisions are possible between "1" and "2" than between "7" and "8,"

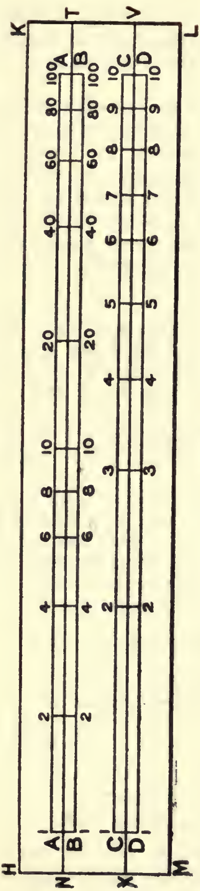

FIG. 4.

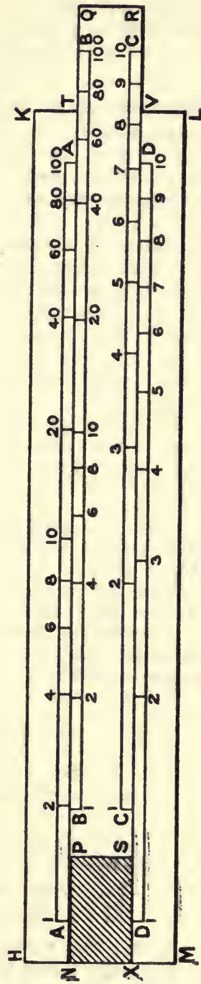

FIG. 5.

FIG. 6.

and, *in general,* the subdivisions are of different value at different parts of the scale.

Note also that the graduations at either end of either scale are called the "indices" of that scale.

The scales **AA**, **BB** are graduated on the same principle as **CC** and **DD**, but in each case the numbers run from 1 to 100 instead of from 1 to 10.

132. On Multiplication and Division, with the Slide projecting to the Right.—In order to multiply two numbers together, we may use either the two scales **C** and **D**, or the two scales **A** and **B**.

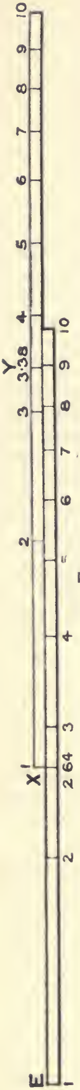

FIG. 7.

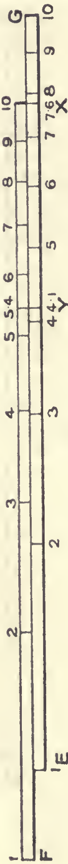

FIG. 8.

Suppose, for example, that we wish to multiply 2·64 by 3·38. Fig. 7 explains the process. We move the slide till the graduation 1 on **C** coincides with the graduation 2·64 on **D** (see the point marked **X** in the figure). Then it will be found that the graduation 3·38 on **C** coincides with the graduation 8·92 on **D** (see the point marked **Y** in the figure); 8·92 is the required product (correct to three significant figures).

EXPLANATION.—Since the point **X** on the scale **D** is graduated as 2·64, therefore the length **EX** represents log 2·64 (cf. § 131); and since the point **Y** on the scale **C** is graduated as 3·38, therefore the length **XY** represents log 3·38.

Thus the length **EY** represents (log 2·64 + log 3·38), which is equal to log (2·64 × 3·38). (Cf. Formula (1), Chapter XIII.)

Since the length **EY** represents log (2·64 × 3·38), therefore the graduation at **Y** is equal to (2·64 × 3·38).

It is obvious from this explanation that this is a mechanical method of obtaining the logarithm of the product by adding the logarithms of the numbers.

The process is expressed in the following rule:—

Rule 1.—To obtain the product $p \times q$—

Set 1 on C, to p on D;
at q on C, find the product on D.

The rule is exactly the same if we are using the scales **A** and **B**, viz.—

Set 1 on **B** to p on **A**;
at q on **B**, find the product on **A**.

Using the scales A and B, we can work up to 100; but using C and D the results are easier to read, as the graduations are larger.

Since division is the reverse of multiplication, this operation can be performed by an obvious modification of the preceding rule.

Rule 2.—To divide *r* by *s*—

Set *s* on C to *r* on D;
at 1 on C, find the quotient on D.

Or, set *s* on B to *r* on A; at 1 on B, find the quotient on A.

EXAMPLE.—*Divide* 8·92 *by* 3·38.

In Fig. 7 we have set 3·38 on C to 8·92 on D; then 1 on C coincides with 2·64 on D, which is therefore the required quotient.

EXPLANATION.—We have already shown that $2·64 \times 3·38 = 8·92$; it follows that $8·92 \div 3·38 = 2·64$.

Note that where a point on the scale falls between two consecutive graduations, its value can (with practice) be estimated correctly by the eye.

EXAMPLES.—XCII.

1. Using the scales C and D, evaluate—
(i.) $3·12 \times 2·24$. (ii.) $1·89 \times 4·25$. (iii.) $3·1 \times 2·36$.
(iv.) $2·88 \times 3·16$. (v.) $8·72 \times 1·12$. (vi.) $1·22 \times 7·3$.

2. Using the scales C and D, evaluate—
(i.) $4·58 \div 2·36$. (ii.) $6·23 \div 5·72$. (iii.) $3·22 \div 2·12$.
(iv.) $9·55 \div 3·22$. (v.) $10 \div 3·14$. (vi.) $10 \div 8·12$.

3. Using the scales A and B, evaluate—
(i.) $8·72 \times 7·28$. (ii.) $12·4 \times 7·22$. (iii.) $15·4 \times 2·58$.
(iv.) $10·5 \times 9·25$. (v.) $5·68 \times 12·5$. (vi.) $12·4 \div 7·22$.
(vii.) $55·4 \div 2·58$. (viii.) $70·5 \div 9·25$. (ix.) $98·5 \div 1·35$.
(x.) $100 \div 87·5$.

133. On Multiplication and Division, with the Slide projecting to the Left.—If, using the scales C and D, we try to multiply 7·6 by 5·4, after setting 1 on C to 7·6 on D, we shall find that 5·4 on C lies beyond the scale D. In this case we use the right-hand index of C in place of the left-hand index, and *multiply the final reading by* 10; that is to say, we set 10 on C to 7·6 on D (Fig. 8). We then find that 5·4 on C coincides with 4·10 on D; thus the required product is $10 \times 4·10 = 41·0$, correct to three significant figures.

EXPLANATION.—In Fig. 8, the logarithm of the required product is the sum of the logarithms of 7·6 and 5·4; this sum is represented by EX + FY. But $EX + FY = EY + YX + FY = EY + FX$; and $EY + FX$ represents $\log 4·10 + \log 10 = \log (4·1 \times 10) = \log 41$. Thus 41 is the required product.

A similar argument will hold for the scales A and B, but in this case, if we use the right-hand index instead of the left-hand index, we must *multiply the result by* 100.

If, then, the rule in the preceding paragraph does not give the product of two given numbers, we use the following rule:—

Rule 3.—To multiply *p* by *q*—

Set 10 on C to *p* on D;
at *q* on C, find the number on D; multiply this number by 10.

Or, set 100 on B to *p* on A;
at *q* on B, find the number on A; multiply this number by 100.

In the same way, if the rule in the preceding paragraph does not give the quotient of two given numbers, we use the following rule:—

Rule 4.—To divide r by s—

Set s on C to r on D;
at 10 on C, find the number on D; divide this number by 10.
Or, set s on B to r on A;
at 100 on B, find the number on A; divide this number by 100.

EXAMPLE.—*Using the scales* A *and* B, *divide* 3·75 *by* 45·5. Set 45·5 on B to 3·75 on A; at 100 on B, we find 8·24 on A; therefore the required quotient is 8·24 ÷ 100, *i.e.* ·0824.

EXAMPLES.—XCIII.

1. Using the scales C and D, evaluate—
   (i.) 5·35 × 2·64. (ii.) 3·45 × 7·85. (iii.) 8·25 × 9·25.
   (iv.) 6·85 × 4·84. (v.) 1·44 × 9·95.
2. Using the scales C and D, evaluate—
   (i.) 2·44 ÷ 5·65. (ii.) 5·65 ÷ 8·75. (iii.) 1·26 ÷ 8·95.
   (iv.) 5·35 ÷ 6·62. (v.) 2·48 ÷ 9·15.
3. Using the scales A and B, evaluate—
   (i.) 5·25 × 29·5. (ii.) 3·75 × 45·6. (iii.) 45·6 ÷ 68·5.
   (iv.) 3·2 ÷ 47. (v.) 23·5 × 67·5. (vi.) 89·5 × 81.
   (vii.) 33·4 × 43·5. (viii.) 2·24 ÷ 82·5. (ix.) 7·35 ÷ 45·5.
   (x.) 91·5 ÷ 99·5.

134. Multiplication and Division of any Two Numbers.

Rule 5.—To multiply or divide any two numbers of three significant figures: in each number place the decimal point after the first significant figure; multiply or divide the two numbers so obtained by the slide rule; the result gives the significant figures of the required product or quotient.

Determine the correct position of the decimal point by a rough calculation.

EXAMPLE (1).—*Multiply* ·00234 *by* ·0785.

Placing the decimal point after the first significant figure, we obtain 2·34 and 7·85. Set 10 on C to 2·34 on D; we find that 7·85 on C coincides with 1·84 on D.* Thus the significant figures of the required product are 184. But the required product is roughly ·002 × ·08 = ·00016. Thus we interpret the result as ·000184.

EXPLANATION.—The product ·00234 × ·0785 differs from the product 2·34 × 7·85 in the position of the decimal point only. Thus, since 184 are the significant figures of the latter product, they are also the significant figures of the former product.

EXAMPLE (2).—*Multiply* ·0348 *by* 224,000.

Multiplying 3·48 by 2·24 we obtain 7·79; ∴ $·0348 \times 224{,}000 = 3·48 \times 10^{-2} \times 2·24 \times 10^5 = 3·48 \times 2·24 \times 10^{-2} \times 10^5 = 7·79 \times 10^3 = 7790$. (Cf. § 104.)

Note the different methods shown in Examples (1) and (2) for determining the position of the decimal point in the final result.

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* Note that we do not multiply by 10 (as we strictly should, in accordance with the preceding paragraph), because this does not affect the significant figures.

EXAMPLE (3).—*Multiply* 3·2483 *by* 215·27 *correct to three significant figures.*
We multiply 3·25 by 215, taking each of the given quantities correct to three significant figures.* Proceeding as before, we obtain 6990.

EXAMPLE (4).—*Divide* 2·8532 *by* 435·25 *correct to three significant figures.*
We divide 2·85 by 4·35; the result is ·0655. Thus the significant figures of the required quotient are 655. But the required quotient is roughly 2·8 ÷ 400 = ·007; thus we interpret the result as ·00655.

EXAMPLES.—XCIV.

[In these examples it is better to use only the scales **C** and **D**.]
Evaluate correctly to three, or (if you can) to four, significant figures—

1. 32·5 × ·725. **2.** 1240 × 3650. **3.** 3250 ÷ ·725.
4. 1340 ÷ ·365. **5.** ·00785 × ·00124. **6.** ·000125 × ·0000117.
7. ·00785 ÷ ·000124. **8.** ·00126 ÷ 23·8.
9. 335,782 × 228,614. **10.** ·00004582 × 23·2791.
11. 6,821,400 ÷ 327,800,000. **12.** ·0000028527 ÷ ·00012689.

135. To evaluate $\frac{p \times q}{h}$.

Rule 6.—To evaluate $\frac{p \times q}{h}$. In each quantity place the decimal point after the first significant figure.

Set h on **C** to p on **D**;
at q on **C**, find the significant figures of the result on **D**.

EXAMPLE (1).—*Evaluate* $\frac{34·5 \times 67·5}{2620}$.

Set 2·62 on **C** to 3·45 on **D**; at 6·75 on **C**, we find 8·89 on **D**.

Thus $\frac{34·5 \times 67·5}{2620} = \frac{3·45 \times 10 \times 6·75 \times 10}{2·62 \times 10^3} = \frac{3·45 \times 6·75}{2·62} \times 10^{-1}$
$= 8·89 \times 10^{-1} = ·889$

EXPLANATION.—When we set 2·62 on **C** to 3·45 on **D**, then 1 on **C** indicates on **D** the quotient (3·45 ÷ 2·62). (See Rule 2.)

But since in this position of the slide, 1 on **C** is set to (3·45 ÷ 2·62) on **D**, then 6·75 on **C** indicates on **D** the product (3·45 ÷ 2·62) × 6·75. (See Rule 1.)

If we try to apply this rule to evaluate $\frac{1·55 \times 2·26}{4·05}$, we shall find that after setting 4·05 on **C** to 1·55 on **D**, then 2·26 on **C** projects beyond the scale **D**. To meet this difficulty we use the following rule :—

Rule 7.—Whenever we wish to use a graduation on one scale which projects beyond the other scale—

Set the runner † to one index of C (or B);
set the other index of C, or B, to the runner.

* Note that if the fourth significant figure in each of the given quantities is 5 or 6, it will probably be more correct to increase the third significant figure by 1 in one quantity, but not in the other.

Thus 42·852 × ·34763 should be taken as 42·8 × ·348, not 42·9 × ·348.

† The runner consists of a piece of glass, mounted in a frame, which slides along the ruler; on the glass is marked one clear line at right angles to the length of the ruler, which stretches across all four scales; thus we can set this line on the runner to any graduation on any scale.

We then proceed with the process in hand, and the significant figures of the result will not be affected.

EXAMPLE (2).—*Evaluate* $\frac{1\cdot55 \times 2\cdot26}{4\cdot05}$.

Set 4·05 on **C** to 1·55 on **D**;
(2·26 on **C** projects beyond **D**);
set runner to 10 on **C**; set 1 on **C** to runner;
at 2·26 on **C**, we find 8·65.

Thus 865 are the required significant figures; and by the usual method we find that this result must be interpreted as ·865.

EXPLANATION.—By setting the runner to 10 on **C**, and then setting 1 on **C** to the runner, we are moving the slide a distance which represents log 10; and are therefore multiplying the result by 10, which does not affect the significant figures.

EXAMPLES.—XCV.

Evaluate—

1. $\frac{\cdot783 \times 147}{195}$. **2.** $\frac{6\cdot54 \times 42\cdot6}{32\cdot5}$. **3.** $\frac{26\cdot2 \times 32\cdot5}{558}$.

4. $\frac{3\cdot38 \times \cdot426}{94\cdot5}$. **5.** $\frac{228 \times \cdot0236}{6\cdot46}$. **6.** $\frac{8\cdot75 \times 5\cdot25}{32\cdot3}$.

7. $\frac{\cdot0228 \times 12\cdot7}{\cdot0356}$. **8.** $\frac{905 \times \cdot0538}{216}$.

136. On a Series of Multiplications and Divisions.—The slide rule enables us to evaluate very rapidly any fraction whose numerator and denominator each consist of a series of factors, when the numerator contains one more factor than the denominator.

Rule 8.—To evaluate $\frac{p \times q \times r \times s \times t}{h \times k \times l \times m}$.

In each of the given quantities place the decimal point after the first significant figure;

Set h on C to p on D;
runner to q on C; k on C to runner;
runner to r on C; l on C to runner;
runner to s on C; m on C to runner;
at t on C, find on D the significant figures of the required result.

EXAMPLE (1).—*Evaluate* $\frac{3\cdot45 \times 673 \times 28\cdot6 \times 8\cdot75 \times 2\cdot5}{275 \times 9550 \times 6\cdot25 \times 122}$.

In accordance with the above rule, we perform the following processes:—

Set 2·75 on **C** to 3·45 on **D**;
runner to 6·73 on **C**; 9·55 on **C** to runner;
runner to 2·86 on **C**; 6·25 on **C** to runner;
runner to 8·75 on **C**; 1·22 on **C** to runner;
at 2·5 on **C** we find 7·25 on **D**.

Thus the significant figures of the required result are 725.

Also, working the problem very roughly, we obtain—

$$\frac{3 \times 700 \times 30 \times 9 \times 2}{300 \times 10000 \times 6 \times 100} = \frac{63}{100000} = \cdot00063$$

We infer that in the correct result the first significant figure will come in the fourth decimal place; hence the result is ·000725.

EXPLANATION.—By Rule 6, the first position of the runner indicates on **D** the result of the process $\frac{3·45 \times 673}{275}$; it follows, by a second application of Rule 6, that the second position of the runner indicates the result $\left(\frac{3·45 \times 673}{275}\right) \times \frac{28·6}{9550}$ (for we have set 9·55 on **C** to the first result on **D**, and therefore 2·86 on **C** indicates on **D** the significant figures of $\left.\frac{\text{the first result} \times 28·6}{9950}\right)$; in a similar way it can be seen that the remaining steps merely repeat this application of Rule 6.

It is frequently necessary to use Rule 7 in conjunction with Rule 8.

EXAMPLE (2).—*Evaluate* $\frac{33·6 \times 52·5 \times 855 \times ·326}{2·84 \times 473 \times 62·5}$.

Set 2·84 on **C** to 3·36 on **D**;
runner to 5·25 on **C**; 4·73 on **C** to runner;
(8·55 on **C** projects beyond **D**);
(∴ runner to 1 on **C**; 10 on **C** to runner);
runner to 8·55 on **C**; 6·25 on **C** to runner;
(3·26 on **C** projects beyond **D**);
(∴ runner to 10 on **C**; 1 on **C** to runner);
at 3·26 on **C** we find 5·86.

Thus the required significant figures are 5·86.

Working roughly, we obtain—

$$\frac{30 \times 50 \times 900 \times ·3}{3 \times 500 \times 60} = 4·5$$

Thus the first significant figure is in the units' place; and the final result is 5·86.

EXAMPLES.—XCVI.

Evaluate—

1. $\frac{3·25 \times 20·8 \times 36·6 \times 78·4}{5·75 \times 29·5 \times 53·6}$.
2. $\frac{6·46 \times 3·26 \times 57·5 \times 283}{8·55 \times 296 \times 36·6}$.
3. $\frac{785 \times 293 \times 20·7 \times 43·4}{248 \times 1250 \times 137}$.
4. $\frac{24·5 \times 22·2 \times 136 \times 305}{5760 \times 825 \times 7·24}$.
5. $\frac{3248 \times 2476 \times 78·375 \times 65·823 \times 88·275}{92·5 \times 83225 \times 12·227 \times 615}$.

137. To evaluate any Expression involving only Multiplication and Division.—*Any operation* which involves only multiplications and divisions can be performed by Rule 8.

EXAMPLE (1) —*Evaluate* $\frac{32·5 \times 550 \times 15·8}{295}$.

This is equal to $\frac{32·5 \times 550 \times 15·8}{295 \times 1}$.

Applying Rule 8, we proceed thus—

Set 2·95 on C to 3·25 on D;
runner to 5·5 on C; 1 on C to runner;
at 1·58 on C we find 9·58 on D.

Working roughly, $\frac{30 \times 500 \times 16}{300} = 800$; hence the result is 958.

EXAMPLE (2).—*Evaluate* $32{\cdot}6 \times {\cdot}0258 \times 7{\cdot}32 \times 6{\cdot}28 \times {\cdot}00234$.

This is equal to $\frac{32{\cdot}6 \times {\cdot}0258 \times 7{\cdot}32 \times 6{\cdot}28 \times {\cdot}00234}{1 \times 1 \times 1 \times 1}$.

Also, when more convenient, we may regard any factor in the denominator as 10 instead of 1 without affecting the significant figures of the result. Thus, by Rule 8—

Set 1 on **C** to 3·26 on **D** ;
runner to 2·58 on **C** ; 10 on **C** to runner ;
runner to 7·32 on **C** ; 10 on **C** to runner ;
runner to 6·28 on **C** ; 1 on **C** to runner ;
at 2·34 on **C** we find 9·05 on **D**.

Whence, in the usual way, we find the result ·0905.

EXAMPLE (3).—*Evaluate* $\frac{2{\cdot}5}{7{\cdot}25 \times 8{\cdot}48}$.

This is equal to $\frac{2{\cdot}5 \times 1 \times 1}{7{\cdot}25 \times 8{\cdot}48}$. Also, we may replace either the second or third factor in the numerator by 10 without affecting the significant figures. Thus—

Set 7·25 on **C** to 2·5 on **D** ;
runner to 10 on **C** ; 8·48 on **C** to runner
at 10 on **C** we find 4·07 on **D**.

Whence, in the usual manner, we find the result ·0407.

EXAMPLES.—XCVII.

Evaluate—

1. $39{\cdot}6 \times 2{\cdot}58 \times 724$. **2.** $287 \times 36{\cdot}8 \times 5{\cdot}73$.

3. $286 \times {\cdot}0355 \times {\cdot}285$. **4.** $325 \times {\cdot}0525 \times 62{\cdot}6 \times 83{\cdot}5$.

5. $\frac{2{\cdot}54 \times 3{\cdot}72}{8{\cdot}28 \times 1{\cdot}24}$. **6.** $\frac{9{\cdot}25}{345 \times {\cdot}0286}$. **7.** $\frac{1}{3{\cdot}56 \times {\cdot}214 \times 7{\cdot}25}$.

8. $\frac{8{\cdot}75 \times 9{\cdot}3}{12{\cdot}4 \times 3{\cdot}72 \times 8{\cdot}5}$. **9.** $\frac{1}{25{\cdot}8 \times {\cdot}324 \times 5{\cdot}62 \times {\cdot}0248}$.

10. $17{\cdot}3 \times 1410 \times 2{\cdot}85 \times 223$. **11.** $173 \times 2{\cdot}45 \times 145 \times 141$.

12. $\frac{3}{2{\cdot}23 \times 2{\cdot}45}$. **13.** $\frac{582}{246 \times 346}$.

14. $\frac{1}{248 \times 2{\cdot}48 \times {\cdot}0248}$. **15.** $\frac{1}{{\cdot}325 \times {\cdot}0325 \times {\cdot}00325 \times {\cdot}000325}$.

138. On Squares and Square Roots.—If the runner be set to any number on **D**, it will be found to indicate the square of that number on **A** ; for example, if we set the runner to 7 on **D**, it will then indicate 49 on **A**.

EXPLANATION.—In both scales the distances of the graduations from the point 1 are proportional to the logarithms of the numbers marked on the graduations. Now, in **D** the whole length of the scale corresponds to log 10, which is 1 ; and in **A** the whole length of the scale corresponds to log 100, which is 2. Thus, any distance on A represents double the logarithm which the equal distance would represent on **D** ; but by Formula 4, Chapter XIII., twice the logarithm of a number is equal to the logarithm of the square of that number.

We thus obtain the following rules :—

Rule 9.—To square a number *n* which lies between 1 and 10—

Set the runner to *n* on D ;
at the runner on A find the required square.

Rule 10.—To find the square root of a number *n* which lies between 1 and 100—

Set the runner to *n* on A ;
at the runner on D find the required square root.

EXAMPLE (1).—*Evaluate* $\sqrt{54}$.

Set the runner to 54 on A ;
at the runner on D we find 7·35.

Rule 11.—To find the square of a number which does not lie between 1 and 10—

(i.) Place the decimal point after the first significant figure, noting how many places the point is moved, and in which direction.

(ii.) Square the number so obtained by Rule 9.

(iii.) In this result, move the decimal point twice as many places as it was moved in step (i.), and in the opposite direction.

EXAMPLE (1).—*Evaluate* $(\cdot 00325)^2$.

(i.) In order to place the decimal point after the first significant figure, we must move it three places to the right. We then obtain 3·25.

(ii.) Set the runner to 3·25 on D ; at the runner on A we find 10·6 (correct to three significant figures), or 10·56 (correct to four significant figures).

(iii.) In step (i.) we moved the decimal point three places to the right ; hence in this result we move the decimal point six places to the left ; which gives the answer—

·00001056

EXPLANATION.—
$$\cdot 00325 = 3\cdot 25 \times 10^{-3}$$
$$\therefore (\cdot 00325)^2 = (3\cdot 25)^2 \times 10^{-6}$$
$$= 10\cdot 56 \times 10^{-6} = \cdot 00001056$$

EXAMPLE (3).—*Evaluate* $(23700)^2$.

(i.) In order to place the decimal point after the first significant figure, we move it four places to the left ; * we then obtain 2·37.

(ii.) Set the runner to 2·37 on D ; at the runner on A we find 5·62.

(iii.) In step (i.) we moved the decimal point four places to the left ; hence in this result we must move the decimal point eight places to the right ; which gives the answer—

562,000,000

EXPLANATION.—
$$23{,}700 = 2\cdot 37 \times 10^4$$
$$\therefore (23{,}700)^2 = (2\cdot 37)^2 \times 10^8 = 5\cdot 62 \times 10^8 = 562{,}000{,}000$$

Rule 12.—To extract the square root of a number which does not lie between 1 and 100—

(i.) Alter the number to one which does lie between 1 and 100 by moving the decimal point *some even number of places.*

* Since the decimal point occurs immediately after the units' digit.

Note the number of places and the direction in which the decimal point is moved.

(ii.) Extract the square root of this number by Rule 10.

(iii.) In this result move the decimal place through one-half as many places as it was moved in step (i.), and in the opposite direction.

EXAMPLE (4).—*Evaluate* $\sqrt{\cdot 0000277}$.

(i.) Moving the decimal point six places to the right, we obtain 27·7.

(ii.) Set the runner to 27·7 on **A**; at the runner on **D** we find 5·26.

(iii.) In step (i.) we moved the decimal point six places to the right; hence in this result (5·26) we move the decimal point three places to the left, which gives the answer ·00526.

EXPLANATION.— $\cdot 0000277 = 27\cdot 7 \times 10^{-6}$

$$\therefore \sqrt{\cdot 0000277} = \sqrt{27\cdot 7} \times 10^{-3} = 5\cdot 26 \times 10^{-3} = \cdot 00526$$

Contrast this example very carefully with the next.

EXAMPLE (5).—*Evaluate* $\sqrt{\cdot 000277}$.

(i.) Moving the decimal point four places to the right, we obtain the number 2·77.

(ii.) Set the runner to 2·77 on **A**; at the runner on **D** we find 1·664.

(iii.) In step (i.) we moved the decimal point four places to the right; thus, in this result (1·664) we must move the decimal point two places to the left, which gives the answer ·01664.

EXPLANATION.— $\cdot 000277 = 2\cdot 77 \times 10^{-4}$

$$\therefore \sqrt{\cdot 000277} = \sqrt{2\cdot 77} \times 10^{-2} = 1\cdot 664 \times 10^{-2} = \cdot 01664$$

EXAMPLE (6).—*Evaluate* $\sqrt{322000}$.

(i.) Moving the decimal point four places to the left, we obtain 32·2.

(ii.) By Rule 10, we find $\sqrt{32\cdot 2} = 5\cdot 67$.

(iii.) Moving the decimal point two places to the right, we obtain the answer 567.

EXAMPLES.—XCVIII.

Evaluate—

1. $(3\cdot 78)^2$.	**2.** $(5\cdot 67)^2$.	**3.** $(8\cdot 26)^2$.	**4.** $\sqrt{8\cdot 2}$.
5. $\sqrt{20\cdot 5}$.	**6.** $\sqrt{57\cdot 6}$.	**7.** $(\cdot 0333)^2$.	**8.** $(35\cdot 5)^2$.
9. $(355)^{\frac{1}{2}}$.	**10.** $(3333)^{\frac{1}{2}}$.	**11.** $(3270)^2$.	**12.** $(\cdot 00058)^{\frac{1}{2}}$.
13. $(\cdot 000058)^{\frac{1}{2}}$.	**14.** $(33500)^2$.	**15.** $(33500)^{\frac{1}{2}}$.	**16.** $(335000)^{\frac{1}{2}}$.

139. On cubing a number which lies between 1 and 10.

Rule 13.—To cube any number *n* which lies between 1 and 4·64;—

Set 1 on C to *n* on D;
at *n* on B find the result on A.

EXAMPLE (1).—*Evaluate* $(3\cdot 26)^3$.

Set 1 on **C** to 3·26 on **D**;
at 3·26 on **B** * we find 34·6 on **A**.

---

* It will be found convenient to mark this position of 3·26 on **B** by means of the runner, to facilitate the reading of the corresponding point on **A**. The runner may often be used with advantage for this purpose in any type of calculation.

EXPLANATION.—Fig. 9 represents the position of the slide. The length **QS** on the scale **D** represents log 3·26; thus the equal length **PR** on the scale **A** represents 2 log 3·26 (see § 138). Also the length **RT** on the scale **B** represents log 3·26, and therefore also represents log 3·26 on the scale **A**. Thus the whole length **PT** on the scale **A** represents 3 log 3·26, *i.e.* log $(3{\cdot}26)^3$ (by Formula 4, Chapter XIII.). Thus the reading at **T** on the scale **A** gives $(3{\cdot}26)^3$.

Rule 14.—To cube any number *n* which lies between 4·64 and 10—

Set 10 on C to *n* on D;
at *n* on B find the reading on A;
multiply this reading by 100.

EXAMPLE (2).*—*Evaluate* $(7{\cdot}45)^3$.

Set 10 on **C** to 7.45 on **D**;
at 7·45 on **B** we find 4·14 on **A**.
$4{\cdot}14 \times 100 = 414$

EXPLANATION.—Using the index 10 on **C**, instead of the index 1, is equivalent to setting the slide back a distance equal to the whole length of either scale. Since the whole length of the scale **A** represents log 100, the final reading on the scale **A** represents $(7{\cdot}45)^3 \div 100$; *i.e.*

$$(7{\cdot}45)^3 \div 100 = 4{\cdot}14$$

whence $(7{\cdot}45)^3 = 4{\cdot}14 \times 100 = 414$.

EXAMPLES.—XCIX.

Evaluate—

1. $(1{\cdot}58)^3$.	**2.** $(2{\cdot}56)^3$.	**3.** $(2{\cdot}18)^3$.	**4.** $(7{\cdot}26)^3$.
5. $(5{\cdot}55)^3$.	**6.** $(8{\cdot}88)^3$.	**7.** $(4{\cdot}15)^3$.	**8.** $(2{\cdot}98)^3$.

140. On cubing a number which does not lie between 1 and 10.

Rule 15.—To cube a number which does not lie between 1 and 10—

(i.) Place the decimal point after the first significant figure, noting how many places to right or left we are moving the point.

(ii.) Cube the number so obtained by Rule 13 or 14.

(iii.) In this result move the decimal point three times as many places as in (i.), and in the opposite direction.

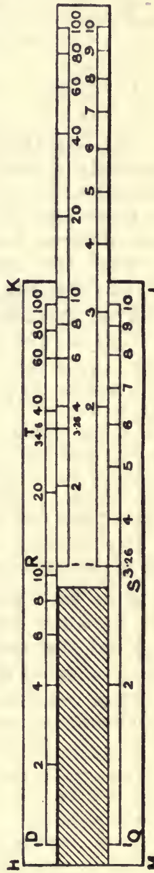

FIG. 9.

EXAMPLE (1).—*Evaluate* $(326)^3$.

(i.) By moving the point two places to the left, we obtain 3·26.

(ii.) By applying Rule 13, we find $(3{\cdot}26)^3 = 34{\cdot}6$.

(iii.) In (i.) we moved the point two places to the left; thus, in this result we must move the point six places to the right. This gives 34,600,000.

* If we try to use Rule 13, we shall find that 7·45 on **B** projects beyond the scale **A**.

EXPLANATION.—$326 = 3{\cdot}26 \times 10^2$; thus—

$$(326)^3 = (3{\cdot}26)^3 \times 10^6 = 34{\cdot}6 \times 10^6 = 34{,}600{,}000$$

EXAMPLE (2).—*Evaluate* $({\cdot}745)^3$.

(i.) By moving the decimal point one place to the right, we obtain 7·45.

(ii.) By Rule 14, $(7{\cdot}45)^3 = 414$.

(iii.) In this result we move the point three places to the left, which gives ·414.

EXAMPLES.—C.

Evaluate—

1. $(15{\cdot}8)^3$.	**2.** $(256)^3$.	**3.** $({\cdot}218)^3$.	**4.** $({\cdot}0726)^3$.
5. $(55{\cdot}5)^3$.	**6.** $({\cdot}888)^3$.	**7.** $({\cdot}0415)^3$.	**8.** $(298)^3$.

141. On Cube Roots.—The operation of extracting a cube root is obviously the reverse of that of forming the cube; hence we require to reverse the operations of the last paragraph.

Rule 16.—To extract the cube root of a number n which lies between 1 and 100—

Find the position of the slide in which

the reading on B at n on A = the reading on D at 1 on C.

This reading on B or D is then the required cube root.

This is obviously the reverse of Rule 13.

EXAMPLE (1).—*Evaluate* $\sqrt[3]{34{\cdot}6}$.

It will be found by repeated trial that there is only one position of the slide in which

the reading on **B** at 34·6 on **A** * = the reading on **D** at 1 on **C**.

This will be the position represented in Fig. 9—

Where the reading on **B** at 34·6 on **A** is 3·26;
and also the reading on **D** at 1 on **C** is 3·26.

Hence 3·26 is the required cube root.

EXPLANATION.—In Example (1), § 139, we have shown that this same position of the slide indicates that $(3{\cdot}26)^3 = 34{\cdot}6$. It follows that $\sqrt[3]{34{\cdot}6} = 3{\cdot}26$.

Rule 17.—To evaluate the cube root of a number n which lies between 100 and 1000—

Find the position of the slide in which

the reading on B at $(n \div 100)$ on A = the reading on D at 10 on C.

This reading is the required cube root.

This is the reverse of Rule 14.

EXAMPLE (2).—*Evaluate* $\sqrt[3]{396}$.

Here $396 \div 100 = 3{\cdot}96$.

It will be found by repeated trial that there is only one position of the slide in which

the reading on **B** at 3·96 on **A** = the reading on **D** at 10 on **C**.

Each of these readings is then 7·34, which is therefore the required cube root.

Rule 18.—To evaluate the cube root of a number which does not lie between 1 and 1000—

(i.) Convert the given number into a number which does lie

* It will be found very convenient to mark the position of 34·6 on **A** by means of the runner.

between 1 and 1000, by moving the decimal point. The number of places through which the decimal point is moved must be 3 or a multiple of 3. Note the number of places through which the decimal point is moved, and the direction.

(ii.) Obtain the cube root of the new number by the use of Rule 16 or Rule 17.

(iii.) In this result move the decimal point one-third as many places as in step (i.), and in the opposite direction.

EXAMPLE (3).—*Evaluate* $\sqrt[3]{31200000}$.

(i.) We can convert this into a number which lies between 1 and 1000 in three ways, viz.—

We convert it to 312 by moving the point five places to the left.
,, ,, 31·2 ,, ,, six ,, ,,
,, ,, 3·12 ,, ,, seven ,, ,,

But, in accordance with the rule, the number of places through which the point is moved must be a multiple of 3. Thus we must move the point six places to the left, which gives the number 31·2.

(ii.) By Rule 16 we find $\sqrt[3]{31{\cdot}2} = 3{\cdot}15$.

(iii.) In step (i.) we moved the point six places to the left; hence in this result (3·15) we must move the point two places to the right. This gives the answer 315.

EXPLANATION.—

$$31200000 = 31{\cdot}2 \times 10^6$$
$$\therefore \sqrt[3]{31200000} = \sqrt[3]{31{\cdot}2} \times 10^2 = 3{\cdot}15 \times 10^2 = 315$$

Contrast this with Example (4).

EXAMPLE (4).—*Evaluate* $\sqrt[3]{312000}$.

(i.) We can convert this into a number which lies between 1 and 1000 by moving the decimal point either three, four, or five places to the left. We therefore move it three places to the left, obtaining the number 312.

(ii.) By Rule 17 we find $\sqrt[3]{312} = 6{\cdot}78$.

(iii.) In step (i.) we moved the decimal point three places to the left. Thus in this result (6·78) we must move the decimal point one place to the right. This gives the answer 67·8.

EXAMPLE (6).—*Evaluate* $\sqrt[3]{{\cdot}00000004}$.

(i.) In ·00000004, if we move the point nine places to the right, we obtain the number 40.

(ii.) By Rule 16, $\sqrt[3]{40} = 3{\cdot}42$.

(iii.) From this result, moving the decimal point three places to the left, we obtain ·00342.

EXPLANATION.—

$${\cdot}00000004 = 40 \times 10^{-9}$$
$$\therefore \sqrt[3]{{\cdot}00000004} = \sqrt[3]{40} \times 10^{-3} = 3{\cdot}42 \times 10^{-3} = {\cdot}00342$$

EXAMPLES.—CI.

Evaluate the cube roots of—

1. 28. **2.** 36. **3.** 90. **4.** 280.
5. 360. **6.** 820. **7.** 10. **8.** 200.
9. 350. **10.** 81. **11.** 34,500. **12.** 67,000,000.
13. 8,410,000. **14.** ·0257. **15.** ·8. **16.** ·000052.
17. 863,721,000. **18.** ·935. **19.** ·0000008. **20.** ·000495.
21. 90,200. **22.** ·902.

142. Miscellaneous Calculations.—Convenient combinations of the preceding rules can be found in many instances.

EXAMPLE (1).—*Evaluate* $\frac{3{\cdot}45 \times (87{\cdot}5)^2 \times 725}{{\cdot}65 \times {\cdot}084}$.

Here we may, of course, evaluate $(87{\cdot}5)^2$ by Rule 11; and, using this result, we can then evaluate the given expression by Rule 8. But we can obtain the required result rather more easily by using Rule 8, and working with the scales **A** and **B**, but using scale **C** * instead of **B** for the factor which is to be squared. Thus—

Set 6·5 on **B** to 3·45 on **A**;
runner to 8·75 on **C**, 8·4 on **B** to runner;
at 7·25 on **B** we find 35·1 on **A**.

The required answer is 351,000,000.

A third method, which is at least equally good, would be to write the given expression as $\frac{3{\cdot}45 \times 87{\cdot}5 \times 87{\cdot}5 \times 725}{{\cdot}65 \times {\cdot}084 \times 1}$, and apply Rule 8.

EXAMPLE (2).—*Evaluate* $\frac{285 \times \sqrt{375} \times \sqrt{21} \times 520}{38 \times 125 \times 742}$.

In this case we may evaluate $\sqrt{375}$ and $\sqrt{21}$ by Rule 12 and Rule 10, and then apply Rule 8. But it will be easier to use Rule 8, working with scales **C** and **D**, but using scale **B** † instead of **C** for the factors whose square roots are required, *remembering that in these factors the decimal point is to be moved an even number of places only* (see Rule 12). Thus—

Set 3·8 on **C** to 2·85 on **D**;
runner to 3·75 on **B**, 1·25 on **C** to runner;
runner to 21 on **B**, 7·42 on **C** to runner;
at 5·2 on **C** we find 3·73 on **D**.

The required answer is 3·73.

EXAMPLE (3). $\frac{\sqrt{2{\cdot}73} \times \sqrt{92{\cdot}5}}{\sqrt{3460}}$.

In this case we may use Rule 6, working entirely on the scales **A** and **B**, but reading the result on **D** by means of the runner.

Set 34·6 on **B** to 2·73 on **A**;
at 92·5 on **B** we find 2·7 on **D**.

The required answer is ·27.

EXAMPLES.—CII.

Evaluate—

1. $23{\cdot}6 \times \sqrt{4590}$. **2.** $3{\cdot}8 \times (7{\cdot}2)^3$. **3.** $\frac{3{\cdot}8}{(7{\cdot}2)^3}$.

4. $\frac{(5{\cdot}87)^3}{(6{\cdot}75)^2}$. **5.** $\frac{(2{\cdot}86)^2 \times (7{\cdot}25)^2}{(4{\cdot}48)^3}$. **6.** $\frac{(28{\cdot}6)^2 \times (67{\cdot}5)^2}{(5{\cdot}7)^3}$.

7. $\frac{23{\cdot}6}{\sqrt{4590}}$. **8.** $\frac{293 \times 587}{\sqrt{423000}}$. **9.** $\frac{({\cdot}00358)^3}{({\cdot}00067)^3}$.

10. $\sqrt{\frac{58{\cdot}7 \times 23{\cdot}5}{62{\cdot}5 \times 345}}$. **11.** $\sqrt[3]{\frac{29{\cdot}3 \times 28{\cdot}7}{4150}}$. **12.** $\sqrt[3]{\frac{890 \times 3{\cdot}75}{240 \times {\cdot}172}}$.

---

* Since any graduation on **B** is the square of the corresponding graduation on **C**.

† Since any graduation on **C** is the square root of the corresponding graduation on **B**.

143. On the Inverted Slide.—If the slide be placed in the rule in the reversed position, so that the scale **C** is in contact with the scale **A**, and the scale **B** in contact with the scale **D**, we can still use the scales **C** and **D** in conjunction, by means of the runner; and it will then be found that the *rules for multiplication* in the ordinary position will effect *division* in the present position, and *vice versâ*.

This is explained by the fact that the graduations on **C** now run from right to left, instead of from left to right; thus processes which in the ordinary position effect the addition of logarithms, in this position effect their subtraction, and *vice versâ*.

Some students may find it easier to use the following rules for cube and cube root, which may be learnt instead of Rules 13–18:—

Rule 19.—To cube any number *n*—

Place the decimal point after the first significant figure in *n*.

Invert the slide.

Set *n* on C to *n* on A;

at 1 or 10 on C find the required significant figures on A.

Rule 20.—To extract the cube root of a number *n* which lies between 1 and 100—

Invert the slide.

Set 1 on C to *n* on A;

find that graduation on C which coincides with the equal graduation on A.

This will be the required cube root.

EXAMPLE.—*Evaluate* $\sqrt[3]{80}$.

Set 1 on **C** to 80 on **A**;

then we shall find that—

4·31 on **C** coincides with 4·31 on **A**.

Thus 4·31 is the required cube root.

Rule 21.—To find the cube root of a number *n* which lies between 100 and 1000—

Invert the slide;

set 10 on C to ($n \div 100$) on A;

find that graduation on C which coincides with the equal graduation on A.

This gives the required cube root.

CHAPTER XV.

ON PLANE GEOMETRY.

144. On the Diagonal Scale.—The diagonal scale enables us to measure inches correctly to two decimal places. Any good set of geometrical instruments will contain a diagonal scale, which is usually engraved on the back of the protractor.

On the opposite page is an accurate diagram of a diagonal scale.

To measure inches and decimals of an inch on this scale we use the figures marked on the top of the scale and those on the right-hand side. The figures marked on the bottom and on the left-hand side refer to half-inches and decimals of a half-inch.

To explain the use of the scale, let us take the length included between the two crosses marked on the figure.

Both crosses are on the horizontal line, which is marked 6 on the right-hand side.

The right-hand cross is on a sloping line which reads 5 at the top.

The left-hand cross is on a vertical line which reads 3 at the top.

These figures give the length of this line as 3·56 inches.

Similarly, if we wish to measure a length of 2·74 inches, we measure it

on the horizontal line which is marked 4 on the right,
from the sloping line which is marked 7 at the top,
to the vertical line which is marked 2 at the top.

Also, if we wish to measure a length of 3·47 half-inches, we measure it

on the horizontal line which is marked 7 on the left,
from the sloping line which is marked 4 at the bottom,
to the vertical line which is marked 3 at the bottom.

To measure off any length from the diagonal scale, we use the **dividers**—the pair of compasses each of whose legs terminates in a metal point. To measure the length of a given line, we adjust the dividers to the exact length of the line, and then by actual testing we find the length, on any of the horizontal lines of the scale, which corresponds to it.

EXPLANATION.—The principle of this scale is that the portions of the horizontal lines which are intercepted between the vertical line o8 and the sloping line oA *increase uniformly as we pass from the upper to the lower* *lines.* The distance 8A intercepted on the lowest line is obviously ·1 inch; the distance intercepted on the line just above is ·09 inch; on the next line, ·08; on the next, ·07; etc.—the distance on the "top line but one" being ·01 inch.

Thus, dividing the distance between the two crosses into the three parts, which we may call **BD**, **DC**, and **CE**—

the length **BD** = the length from 3 to 0 on the top line = 3 inches
,, **DC** = ·06 ,,
,, **CE** = the length from 0 to 5 on the top line = ·5 ,,

Thus the whole length **BE** = 3·56 inches.

A very little practice will enable the student to master the use of the diagonal scale.

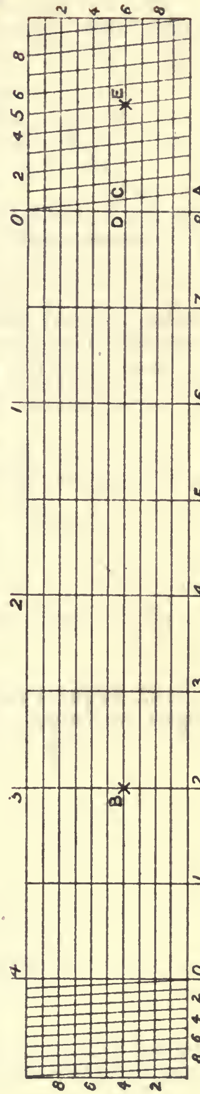

FIG. 10.

EXAMPLES.—CIII.

1. Draw a line about 6 inches long, and from it measure off successive lengths of 1·32, 1·54, 1·48 inches. Test whether the length of the whole distance marked off is accurately equal to the sum of these lengths. (Absolute accuracy is scarcely to be expected.)

2. Draw a line exactly 3·58 inches long; mark off from it a part 1·66 inch long; test whether the remainder is accurately equal to the difference of these lengths.

3. On a line mark off two lengths in succession, each equal to 3·47 half-inches; and test whether their sum is accurately equal to 3·47 inches.

145. On Angles.—By an angle we mean the "corner" formed where two lines meet one another. Thus in Fig. 11 the lines **AB** and **BC**, which meet at **B**, form a "corner" or angle at **B**.

The two lines which form the angle are called the "arms" of the angle, and the point **B** is called the "vertex" or "apex" of the angle, or sometimes the "angular point."

To name an angle we use three letters, of which the *middle letter must denote the angular point*, and the other two letters must denote one point on each arm; thus the first angle in Fig. 11 may be called either the angle **ABC** or the angle **CBA**, the letter **B** being necessarily placed in the middle. Similarly, the second angle in this figure may be called either the angle **DEF** or the angle **FED**; while the third may be called either **GHK** or **KHG**.

146. The size of an angle does not depend on the length of its arms, but merely on its shape at the corner.

Thus in Fig. 12 the angle **ABC** is not equal to the angle **DEF**, although the arms of the first angle are equal to the arms of the

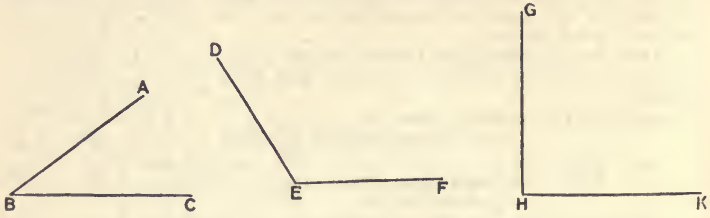

FIG. 11.

second; for the "corner" at **B** would not fit the corner at **E**. But the angle **ABC** is equal to the angle **GHK**; for the corner at **B** will exactly fit the corner at **H**.

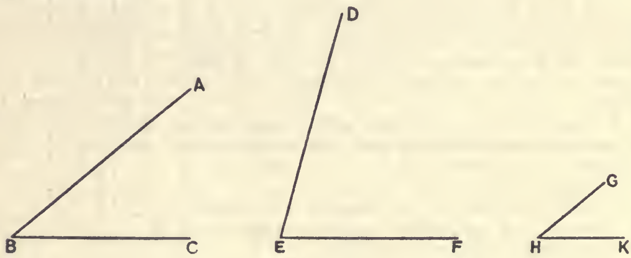

FIG. 12.

147. Right Angles.—When one line "stands on" another line, two angles are formed. Thus in Fig. 13 the line **BD** "stands on" the

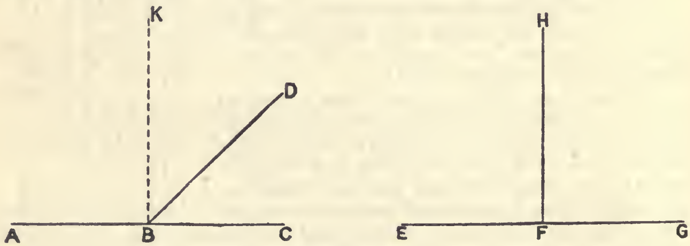

FIG. 13.

line **AC** (*i.e.* it "meets" the line **AC** without passing through it or "cutting it"). Then there are two angles formed at **B**, viz. **ABD** and **CBD**.

If the two angles which are formed by one line standing on another line are equal to one another, each is called a **right angle,** and each of the lines is said to be **perpendicular** to the other. Thus in Fig. 13 the angles **HFE, HFG** are equal; then each is a right angle, and the lines **HF** and **EG** are perpendicular to one another.

Obviously, the angle at each corner of an ordinary sheet of paper is a right angle; the angle of a "set square" is a right angle; the angle between a vertical and a horizontal line is a right angle; etc.

All right angles are equal to one another.

Theorem I.—When one line stands on another line, the two angles thus formed are together equal to two right angles.

This should be obvious; for in Fig. 13, if we draw **BK** perpendicular to **AC**, then the two angles **ABD, DBC** are together equal to the three angles **ABK, KBD, DBC,** which are together equal to the two right angles **ABK, KBC.**

This theorem is proved in Euclid, Book I. Prop. 13.

If two angles are together equal to two right angles, they are said to be **supplementary**; either angle is then called the **supplement** of the other.

If two angles are together equal to one right angle, they are said to be **complementary**; either angle is then called the **complement** of the other.

Thus in Fig. 13 the angles **DBA** and **DBC** are supplementary, and the angles **DBK** and **DBC** are complementary.

148. If two straight lines cut one another they form four angles

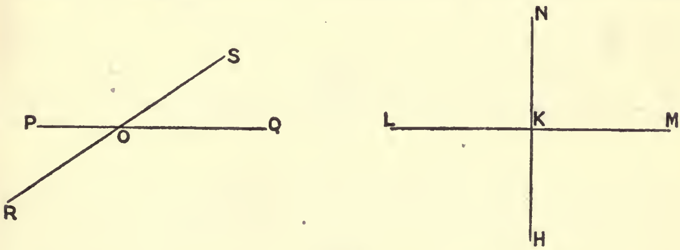

FIG. 14.

at their point of intersection. Thus in Fig. 14 we have four angles formed at the point **O**, viz. **POS, SOQ, QOR, ROP.**

Of these the angle **POR** = angle **SOQ**,
and angle **POS** = angle **QOR**.

This is usually expressed as follows:—

Theorem II.—When two straight lines intersect, the angles at the intersection which are vertically opposite to one another, are equal to one another.

This theorem is proved in Euclid, Book I. Prop. 15.

Notice also that in Fig. 14, since **SO** stands on **PQ**, therefore the angles **POS, SOQ** are together equal to two right angles. Also we may regard the line **QO** as standing on the line **SR**; whence we deduce that the angles **SOQ, QOR** are together equal to two right angles.

Similarly, the angles **POR, ROQ** are together equal to two right angles; and the angles **POR, POS** are together equal to two right angles.

Summing up, we may say that in this figure angles which are vertically opposite are equal, and each pair of adjacent angles are together equal to two right angles.

Theorem III.—If a series of lines be drawn from one point, the sum of the angles thus formed at that point will be together equal to four right angles.

This is illustrated in Fig. 15. A series of lines, **OA, OB, OC, OD,**

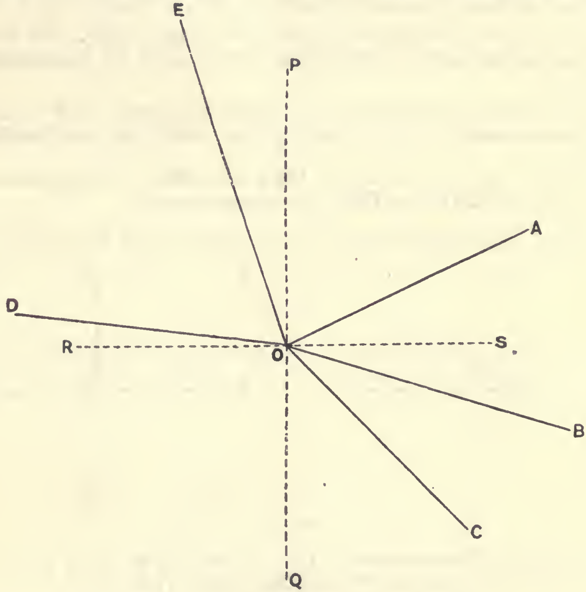

Fig. 15.

OE, have been drawn from **O,** forming the angles **AOB, BOC, COD, DOE, EOA.** But if any pair of perpendicular lines, **PQ, RS,** be drawn through **O,** it is obvious that the sum of the angles mentioned above is equal to the sum of the four angles **POS, SOQ, QOR, ROP,** each of which is a right angle.

Angles which are less than a right angle are called **acute** angles (literally, "sharp" angles). Thus all the angles in Fig. 12 are acute. Also in Fig. 14 the angles **POR** and **SOQ** are acute.

Angles which are greater than a right angle are called **obtuse** angles (literally, "blunt" angles). Thus in Fig. 11 **DEF** is an obtuse angle; and in Fig. 14 **POS** and **QOR** are obtuse angles.

149. On measuring Angles.—To measure angles we use the right angle as a unit. The right angle is divided into 90 equal parts, which are called **degrees**; the degree is divided into 60 equal parts, which are called **minutes**; and the minute into 60 equal parts, which are called **seconds**. The second is thus an exceedingly small angle.

An angle of 20 degrees 14 minutes 10 seconds is represented by the notation $20^\circ\ 14'\ 10''$.

For ordinary geometrical purposes there is no need to use anything smaller than the degree. Any box of geometrical instruments should contain a "protractor," by which angles can be measured in degrees.

150. On the Use of the Protractor.—Fig. 16 is a diagram of a **protractor**. This instrument is usually a rectangular ruler; the lower edge **AB** contains only one mark, notched at its middle point **O**, and rendered more conspicuous by a star. The other three edges are marked with a series of lines notched in the wood, of which

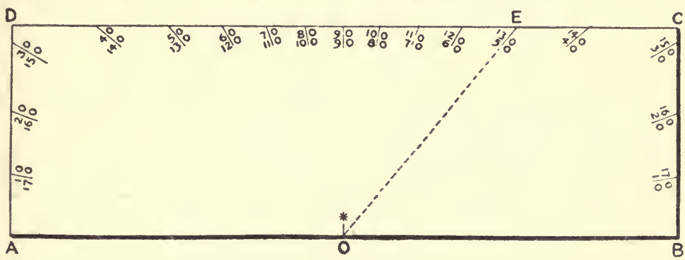

FIG. 16.

only every tenth line is represented in the diagram. The two numbers marked against any notch **E** give the number of degrees in the angles formed by **OE** and **AB**. Thus the outer number 130 indicates that the angle **AOE** is 130°, and the inner number 50 indicates that the angle **EOB** is 50°.

Note that the outer number at any notch refers to the angle contained between the line **OA** and the line which runs from **O** to the notch; while the inner number refers to the angle between the line **OB** and the line which runs from **O** to the notch.

To measure any angle, place the protractor so that **O** is on the angular point, and so that **OA** or **OB** coincides with one arm of the angle, while the other arm passes under the protractor and projects beyond it. Then read the number on the notch which coincides with the latter arm. If the latter arm is too short to project beyond the protractor, it must be produced.

Notice that the sum of the angles AOE, EOA = 130° + 50° = 180° = two right angles, in accordance with Theorem I.

151. Some Geometrical Constructions.—The solutions of the following geometrical problems should be committed to memory by repeated practice :—

(1) *To bisect a given line* AB. (See Fig. 17.)

With centres A and B, and any *equal* radii, describe two circles intersecting at C and D. Join CD, cutting AB at E.

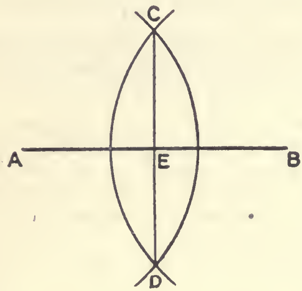

Fig. 17.

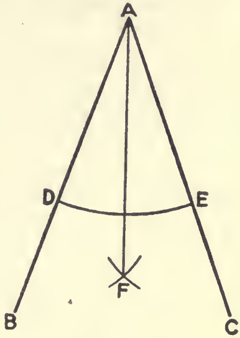

Fig. 18.

Then the point E bisects AB ; *i.e.* AE = EB. (Also the line CD is perpendicular to AB, *i.e.* CD "bisects AB at right angles.")

(2) *To bisect a given angle* BAC. (See Fig. 18.)

With centre A, and any convenient radius, describe a circle cutting AB and AC at D and E respectively. With centres D and E, and any equal radii, describe arcs (*i.e.* portions of circles) intersecting at F. Join AF.

Then AF bisects the angle BAC ; *i.e.* the angle BAF = the angle FAC.

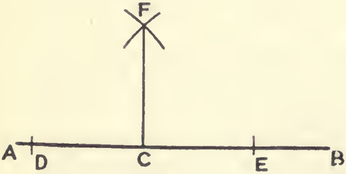

Fig. 19.

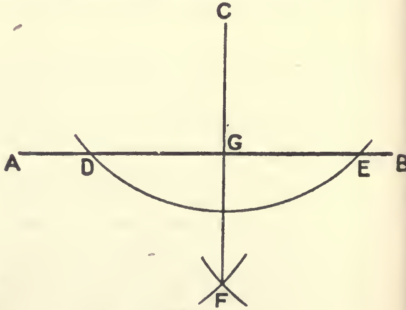

Fig 20.

(3) *At a given point* C *in a given line* AB, *draw a line perpendicular to* AB. (See Fig. 19.)

From C mark off equal portions CD and CE of any convenient length.

With centres **D** and **E**, and any convenient equal radii, describe arcs of circles intersecting at **F**. Join **CF**.

Then **CF** is perpendicular to **AB**.

(4) *From a given point* **C** *without a given straight line* **AB**, *draw a line perpendicular to* **AB**. (See Fig. 20.)

With centre **C**, and any convenient radius, describe a circle cutting **AB** at the points **D** and **E**. With centres **D** and **E**, and any convenient equal radii, describe arcs of circles intersecting at **F**. Join **CF**, cutting **AB** at **G**.

Then the line **CGF** is perpendicular to **AB**.

(5) *At a given point* **C** *in a line* **AB**, *construct an angle equal to a given angle* **DEF**. (See Fig. 21.)

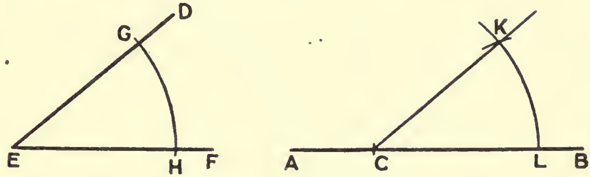

FIG. 21.

With centre **E**, and any convenient radius, describe an arc cutting **ED** and **EF** in **G** and **H** respectively. With centre **C**, and the same radius, describe an arc cutting **CB** at **L**. By means of the compasses measure off on this arc a distance **LK** equal to **HG**. Join **CK**.

Then the angle **KCB** = the angle **DEF**.

The student should practise these constructions and verify the results by actual measurement. For example, in Fig. 20, if we measure the angle **CGB** by the protractor, it will be 90°, since **CG** is perpendicular to **AB**.

152. On Triangles.—A **triangle** is a figure enclosed by three straight lines, such as **ABC** in Fig. 17. It has three "**sides**," **AB**, **BC**, **CA**; and three "**angles**," at **A**, **B**, **C**.

If two sides of a triangle are equal, it is called an **isosceles** triangle.

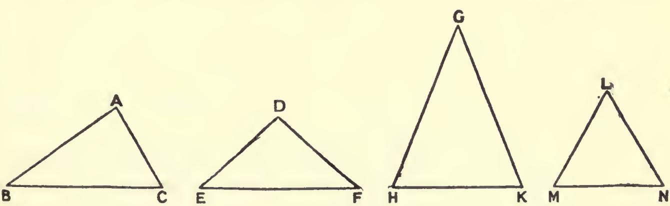

FIG. 22.

For example, the triangles **DEF** and **GHK** (in Fig. 22) are isosceles, because **DE** = **DF** and **GH** = **GK**.

If all three sides of a triangle are equal, it is called an **equilateral** triangle. In Fig. 22 **LMN** is an equilateral triangle.

153. The following are the important theorems with regard to triangles :—

Theorem IV.—If two sides of a triangle are equal, the angles which are opposite to these sides are also equal. [Euclid, I. 5.]

Thus in Fig. 22, since **DE** = **DF**, the angle **DFE** = angle **DEF.**

Theorem V.—All the angles of an equilateral triangle are equal to one another, and are always 60° each.

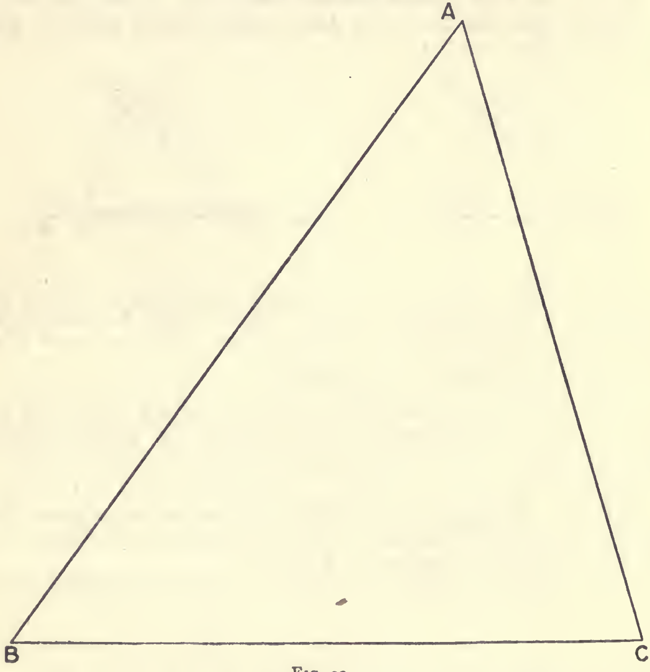

FIG. 23.

Theorem VI.—The three angles of any triangle are together equal to 180°. [Euclid, I. 32.]

Theorem VII.—The greatest angle in a triangle is opposite to the greatest side, and the smallest angle is opposite to the smallest side. [Euclid, I. 18.]

Theorem VIII.—Any two sides of a triangle are together greater than the third. [Euclid, I. 20.]

The truth of Theorems VI., VII., and VIII. is illustrated by Fig. 23. If the sides and angles of this triangle are measured, we shall find—

AB = 4·04 inches, angle **ACB** = 74°
AC = 3·4 „ „ **ABC** = 54°
BC = 3·34 „ „ **BAC** = 52°

Note that since **AC** and **BC** are very nearly equal, the angles opposite to them are very nearly equal.

The student should verify Theorems IV. and V. for himself.

If two triangles are such that the one would fit exactly on to the other, they are said to be **equal in all respects**; such as the triangles **DEF** and **GHK** in Fig. 24. In this case *each side or angle in the one*

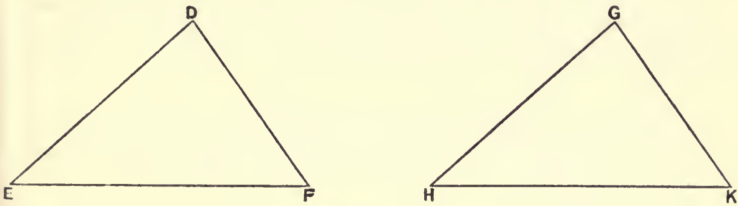

FIG. 24.

triangle is equal to the corresponding side or angle in the other triangle.

154. The following tests for the equality of triangles should be carefully studied :—

(A) If the three sides of one triangle are respectively equal to the three sides of another triangle, the triangles will be equal in all respects. [Euclid, I. 8.]

EXAMPLE.—In Fig. 25, if we know that **AB** = **DE**, **BC** = **EF**, and **CA** = **FD**; then it will follow that the angles **A**, **B**, and **C** are respectively equal to the angles **D**, **E**, and **F**.

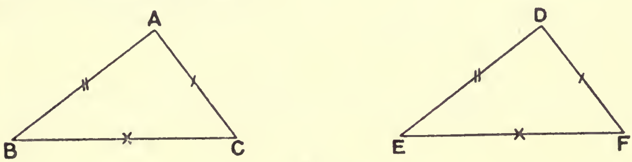

FIG. 25.

(B) If two sides and the included angle in one triangle are respectively equal to two sides and the included angle in another triangle, the triangles will be equal in all respects. [Euclid, I. 4.]

EXAMPLE.—In Fig. 26, if we know that **GH** = **LM**, **GK** = **LN**, and the angle **G** = the angle **L**; then it will follow that the third side **HK** = the third side **MN**, and that the angles **H** and **K** are respectively equal to the angles **M** and **N**.

FIG. 26.

(C) If one side and the adjacent angles in one triangle are respectively equal to one side and the adjacent angles in another triangle, the two triangles will be equal in all respects. [Euclid, I. 26.]

EXAMPLE.—In Fig. 27, if we know that **PQ** = **ST**, and that the angles **P** and **Q** (which are adjacent to the side **PQ**) are respectively equal to the angles **S** and **T** (which are adjacent to the side **ST**); then it will follow that the sides **PR** and **QR** are respectively equal to the sides **SV** and **TV**, and the angle **R** = the angle **V**.

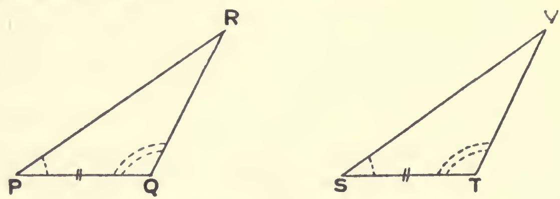

FIG. 27.

(D) If, in one triangle, one side, the angle opposite to it, and one angle adjacent to it, are respectively equal to one side, the angle opposite to it, and one angle adjacent to it in another triangle, then the two triangles will be equal in all respects. [Euclid, I. 26.]

EXAMPLE.—In Fig. 28, if we know that **AB** = **DE**, that the angle **C** (which is opposite to **AB**) is equal to the angle **F** (which is opposite to **DE**),

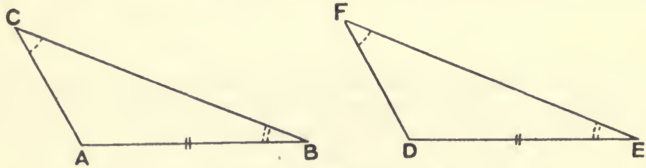

FIG. 28.

and that the angle **B** (which is adjacent to **AB**) is equal to the angle **E** (which is adjacent to **DE**); then it will follow that **BC** = **EF**, **CA** = **FD**, and the angle **A** = the angle **D**.

155. On the Construction of Triangles.

(1) *Two sides of a triangle measure* 1·21 *and* 1·09 *inch respectively; the angle included between these sides measures* 40°. *Construct the triangle.* (See Fig. 29.)

Using the protractor, construct an angle of 40°, **CAB**. Measure **AD** = 1·21 inches, and **AE** = 1·09 inches. Join **ED**.

AED is the required triangle.

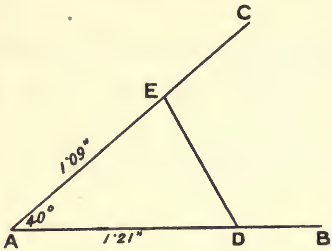

Fig. 29.

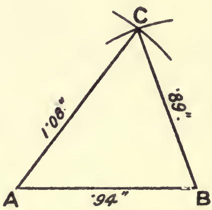

Fig. 30.

(2) *Construct a triangle whose three sides measure* ·94, 1·08, *and* ·89 *inches respectively*. (See Fig. 30.)

Draw a line **AB** of length ·94 inch. Describe two arcs, the first with centre **A** and radius 1·08 inches, the second with centre **B** and radius ·89 inch. **C** is the intersection of these arcs. Join **CA, CB**.

ABC is the required triangle.

N.B.—This problem is impossible if any two of the given lengths are together less than the third. (Cf. Theorem VIII.)

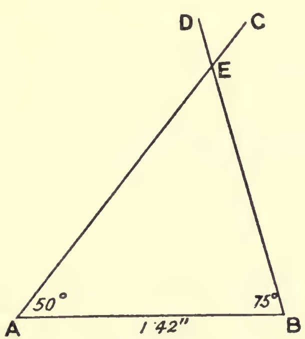

Fig. 31.

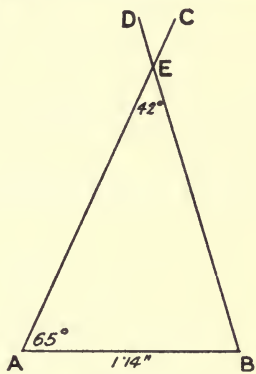

Fig. 32.

(3) *Construct a triangle having one side of length* 1·42 *inch, and the two adjacent angles of magnitude* 50° *and* 75° *respectively.* (See Fig. 31.)

Draw **AB** of length 1·42 inches. Using the protractor, make the angle **BAC** = 50°, and the angle **ABD** = 75°.

ABE is the required triangle.

(4) *Construct a triangle having one side of length* 1·14 *inch, one adjacent angle of magnitude* 65°, *and the angle opposite to this side of magnitude* 42°. (See Fig. 32.)

From 180° subtract the sum of 65° and 42°; result 73°. This gives the third angle of the triangle (Theorem VI.), which will be adjacent to the given side.

Draw **AB** of length 1·14″. Make the angles **BAC** and **ABD** of magnitude 65° and 73° respectively.

ABE is the required triangle.

The last two problems are impossible if the two given angles are together greater than 180° (Theorem VI.).

Note the analogy between these four problems and the four theorems of § 154.

(5) *Construct a triangle having two sides of length* 1·69 *inches and* 1·08 *inches respectively, and having an angle of* 36° *opposite to the latter side.* (See Fig. 33.)

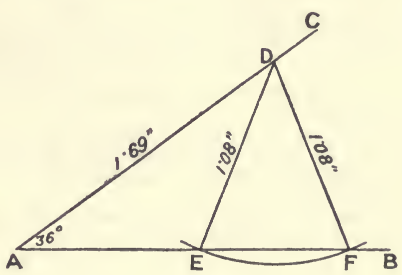

Fig. 33.

Make the angle **CAB** = 36°. Measure off **AD** = 1·69 inches. With centre **D**, and radius 1·08 inches, describe a circle, cutting **AB** at **E** and **F**. Join **DE**, **DF**.

Either of the triangles **ADE**, **ADF**, satisfies the required conditions.

Note that if the circle with centre **D** does not meet **AB**, the problem is *impossible*.

Examples.—CIV.

1. A triangle **ABC** is given by the following data: **AB** = 3·52 inches, **AC** = 2·28 inches, angle **BAC** = 50°. Draw the triangle, and measure the other side and angles.

2. A triangle **PQR** is given by the following data: **PQ** = 3·26 inches, angle **PQR** = 90°, angle **QPR** = 30°. Construct the triangle, and measure the other sides.

3. A triangle **LMN** is given by the following data: **LM** = 3·12 inches, angle **LMN** = 48°, angle **MLN** = 80°. Construct the triangle, and measure the other sides.

4. In a triangle **FGH**, **FG** = 2·58 inches, angle **FGH** = 50°, angle

FHG = 30°. Calculate the angle HFG; then construct the triangle, and measure the side FH.

5. The sides of a triangle are 2, 3, and 4 inches respectively. Construct the triangle, and measure the angles.

6. The sides of a triangle are 5, 6, and 8 inches respectively. Construct the triangle, and measure the angles.

7. Two sides of a triangle measure 1·73 and 1 inch respectively; the angle opposite the latter side is 30°. Construct the triangle, and measure the third side.

In each of the following examples, construct the triangle ABC from the given data, and measure the other sides and angles in each case:—

8. BC = 2, CA = 2·4, AB = 2·8 inches.
9. BC = 1·8, CA = 2·1, AB = 2·4.
10. BC = 1·32, CA = 2·2, AB = 3·08.
11. AB = 1·2, AC = 2, A = 60°.
12. AB = 2·1, AC = 1·5, A = 50°.
13. AB = 2, AC = 2, A = 73°.
14. BC = 2·73, B = 50°, C = 66°.
15. BC = 3·51, B = 40°, C = 61°.
16. BC = 2·28, A = 43°, C = 49°.
17. BC = 2·05, A = 77°, B = 23°.
18. AB = 2, BC = 1·41, A = 30°.
19. AB = 2, BC = 1·25, A = 30°.
20. AB = 1·8, BC = ·9, A = 30°.
21. AB = 1·8, BC = ·9, A = 40°.
22. AC = 2·45, BC = 2, A = 45°.
23. AC = 1·25, BC = 1·41, A = 53°.
24. AB = 1·4, BC = 1, C = 120°.

156. An **acute-angled triangle** is a triangle *each* of whose three angles is acute.

The triangle in Fig. 23 is an acute-angled triangle.

An **obtuse-angled triangle** is a triangle *one* of whose angles is obtuse.

The triangle DEF in Fig. 22 is an obtuse-angled triangle.

A **right-angled triangle** is a triangle *one* of whose angles is a right angle.

No triangle can contain more than one right angle, or more than one obtuse angle. For a right angle contains 90°, and an obtuse angle more than 90°, while the three angles of any triangle together contain only 180°.

In a right-angled triangle, the side opposite the right angle is called the **hypotenuse** of the triangle.

Theorem IX.—The square of the length of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the lengths of the other two sides.

For example, in Fig. 34 the lengths of the sides are as follows:—

$$AB = 3{\cdot}54;\ BC = 2{\cdot}02;\ AC = 1{\cdot}84$$
$$\text{But } 3{\cdot}54^2 = 12{\cdot}5316$$
$$\text{and } (3{\cdot}02)^2 + (1{\cdot}84)^2 = 9{\cdot}1204 + 3{\cdot}3856 = 12\ 5060$$

This is not quite equal to $3{\cdot}54^2$, but it will be found to be greater than $3{\cdot}53^2$; and the apparent discrepancy is due to the fact that we have only measured the lengths correct to two decimal places.

Conversely, if the sides of a triangle are 3, 4, and 5 inches respectively; then, *because* $3^2 + 4^2 = 5^2$ (*i.e.* 9 + 16 = 25), *the angle opposite to the side whose length is* 5 *inches will be a right angle.*

We may also use this theorem to find one side of a right-angled triangle, if we know the other two.

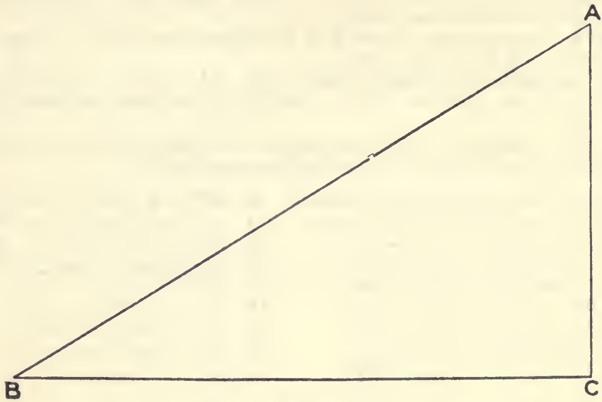

FIG. 34.

EXAMPLE.—*The hypotenuse of a right-angled triangle is* 13 *inches long, and one of the other sides is* 5 *inches long: find the third side.*

Let the third side be x inches long. Then, by Theorem IX.—

$$13^2 = 5^2 + x^2$$

Solving this equation, we obtain—

$$169 = 25 + x^2$$
$$\therefore x^2 = 144$$
$$\text{whence } x = 12 \text{ inches}$$

EXAMPLES.—CV.

1. Construct a triangle **ABC**, having **AB** = 2·28 inches, **AC** = 3·12 inches, angle **BAC** = 90°. Measure the third side, and verify Theorem IX.

2. Calculate the length of the hypotenuse of a right-angled triangle, if the lengths of the other sides are 12 miles and 16 miles respectively.

3. The hypotenuse of a right-angled triangle is 65 yards, and one side is 25 yards: find the other side.

4. Find, both by calculating and by drawing, the hypotenuse of a right-angled triangle, if the lengths of the other sides are 2·24 and 3·36 inches respectively.

5. The hypotenuse of a right-angled triangle is 2·24 inches, and one side is 1·12 inches: find the other side, both by calculating and by drawing. (Cf. Problem (5), § 155, for method of construction.)

In each of the following examples the angle C of the triangle ABC is of magnitude 90°. Construct the triangle from the given data, and measure the other sides and angles:—

6. A = 25°, AC = 2. **7.** A = 40°, BC = 2. **8.** AB = 3, AC = 1·8.
9. AB = 2, AC = 1·73. **10.** AC = 1·5, BC = 2. **11.** B = 40°, CA = 1·5.
12. B = 55°, AB = 2. **13.** A = 35°, AB = 1·6. **14.** A = 20°, AB = 2.

157. On Parallel Lines.

Parallel lines are lines in the same plane, which will never meet, however far they are produced in either direction.

Though there is no difficulty in understanding what we mean by "parallel lines," this definition will repay careful thought.

The lines on an ordinary sheet of ruled foolscap are parallel lines. These lines do not meet *on the paper;* but they are not strictly parallel unless we can assert that, however far we may imagine them produced beyond the paper in either direction, they will never meet.

Again, if on one wall of a room we draw a *vertical* line, and on the opposite wall we draw a *horizontal* line, these two lines would never meet, however far we may imagine them to be produced. But we should obviously not call them parallel lines. They do not satisfy the definition given above, because *they are not in the same plane;* the word "plane" denotes a perfectly flat surface; and it is impossible to conceive any single flat surface which would contain *both* of these lines.

If both lines had been vertical (instead of the one vertical and the other horizontal), then a flat surface *can* be imagined stretching across the room, which would contain the whole length of each of these lines; and since it would still be true that the lines would never meet, they would then be parallel lines.

Theorem X.—Lines in the same plane, which are perpendicular to the same straight line, are parallel to one another.

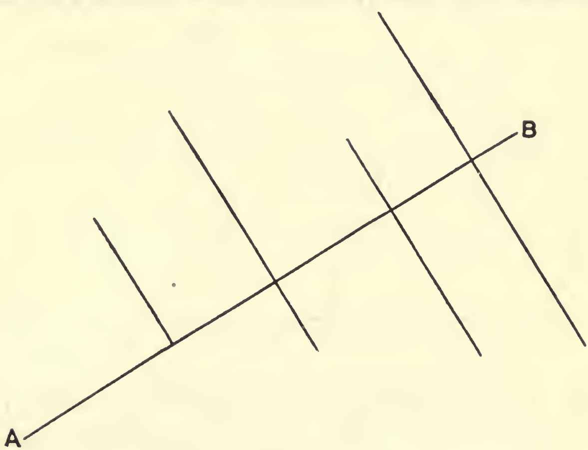

Fig. 35.

Thus in Fig. 35 the other lines are all perpendicular to the line AB, and are parallel to one another.

Theorem XI.—If a straight line cuts two or more parallel

lines, any two of the angles so formed are either equal or supplementary. [Euclid, I. 29.]

For example, in Fig. 36 the straight line PQ cuts three parallel straight lines. All the acute angles marked a are equal; all the

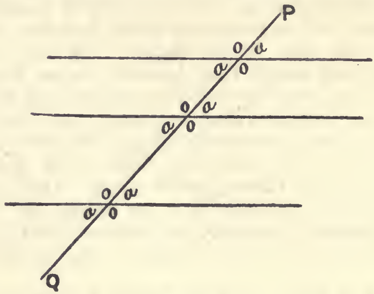

FIG. 36.

obtuse angles marked o are equal; and any angle marked a is supplementary to any angle marked o; that is to say, $a + o = 180°$.

PROBLEM.—*Through a given point* C *draw a line parallel to a given line* AB. (See Fig. 37.)

Take any convenient point D in AB. With centre D, and radius DC, describe an arc cutting AB at E. With centre C, and radius CD, describe an

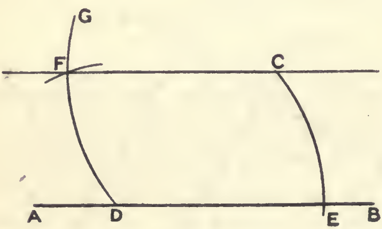

FIG. 37.

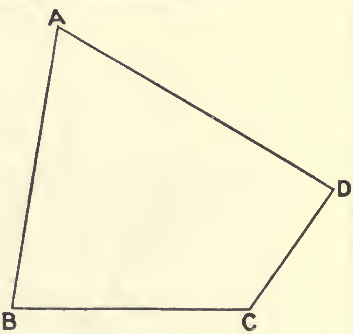

FIG. 38.

arc DG. From the arc DG measure off a portion DF equal to CE. Join CF. CF will then be parallel to AB.

(In practice, it is usually more rapid and more reliable to use marquois scales, a pair of set squares, or a parallel ruler.)

158. On Quadrilaterals.—Any figure bounded by four straight lines is called a **quadrilateral**; such as ABCD in Fig. 38.

Theorem XII.—The four angles of a quadrilateral are together equal to 360°, i.e. to four right angles.

This follows from Theorem VI. For if in Fig. 38 we join AC, we divide the quadrilateral into two triangles.

The student should verify this by drawing a quadrilateral and measuring the angles.

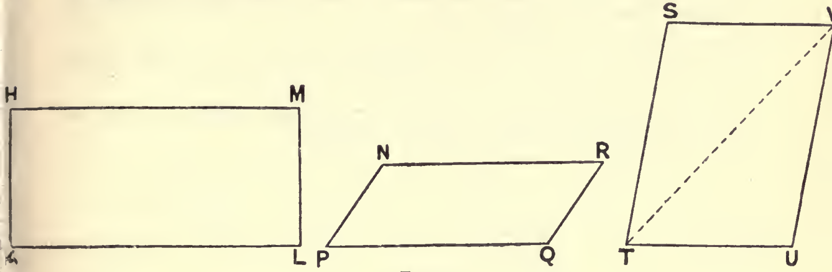

Fig. 39.

A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel; such as **NPQR** in Fig. 39, where **NP** is parallel to **RQ**, and **NR** is parallel to **PQ**; and also **STUV.**

Theorem XIII.—In any parallelogram—

(i.) Opposite sides are equal—

in **NPQR** we have **NP** = **RQ**; **NR** = **PQ**

[Euclid, I. 34.

(ii.) Opposite angles are equal—

in **NPQR** the angle at **N** = the angle at **Q**
and the angle at **P** = the angle at **R**

[Euclid, I. 34.

(iii.) Any pair of adjacent angles are supplementary—

the angle at **N** + the angle at **P** = 180°, etc.

[Euclid, I. 29.

This last statement is obvious from Theorem X., for the line **NP** meets the parallel lines **NR** and **PQ**; hence the angles formed at **N** and **P** are either equal or supplementary; and they are not equal, as one is obtuse and the other acute.

The line which joins either pair of opposite angular points of a quadrilateral is called a **diagonal**; such as **TV** in Fig. 39.

Theorem XIV.—Any diagonal divides a parallelogram into two triangles, which are equal in all respects.

Thus in Fig. 39 the diagonal **TV** divides the parallelogram into two triangles **STV, UVT,** which are equal in all respects; for the three sides **ST, TV, VS** of the one triangle are respectively equal to the three sides **UV, VT, TU** of the other triangle (cf. § 154, A). Thus the lower triangle will fit exactly on to the upper triangle if it is simply turned round and placed with **T** on **V** and **U** on **S.**

The student should verify this with a piece of paper cut in the shape of a parallelogram, and then cut into two pieces along one diagonal.

A **square** is a quadrilateral whose sides are all equal, and whose angles are all right angles.

A **rectangle** is a quadrilateral whose angles are all right angles, and in which opposite sides are equal; such as HKLM in Fig. 39, in which HK = ML, and HM = KL.

A **rhombus** is a quadrilateral whose sides are all equal, but whose angles are not right angles; such as PQRS in Fig. 40.

A square, rectangle, and rhombus *are all parallelograms*, for in each of these figures opposite sides are parallel.

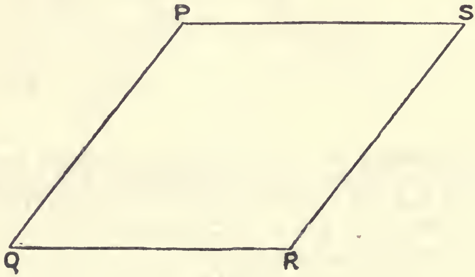

FIG. 40.

Hence Theorems XIII. and XIV. are true for each of these figures.

The student should also verify for himself, by actual drawing, the following properties:—

(i.) **The diagonals of any parallelogram bisect each other**; *i.e.* divide each other into exact halves.

(ii.) **The diagonals of a rhombus bisect each other, and are at right angles.**

(iii.) **The diagonals of a rectangle bisect each other, and are of equal length.**

(iv.) **The diagonals of a square bisect each other, are at right angles, and are of equal length.**

159. On the Construction of Quadrilaterals.—There are several forms of this problem, but the following cases should be sufficient to suggest the general principles of solution:—

(1) *Construct the quadrilateral* ABCD *from the following measurements in inches*: AB = 1·12, BC = ·62, CD = 1·6, DA = 1·49, AC = 1·53. (See Fig. 41. Note that we are given the four sides and one diagonal.)

Draw AC = 1·53. Find the point B by the intersection of two arcs—the first with centre A and radius 1·12; the second with centre C and radius ·62. Find the point D by the intersection of two arcs—the first with centre A and radius 1·49; the second with centre C and radius 1·6.

(2) *Construct the quadrilateral* ABCD *from the following measurements in inches:* **AB** = 1·55, **BC** = ·73, **CD** = 1·03, **DA** = 1·6; angle **DAB** = 53°. (See Fig. 42. Note that we are given the four sides and one angle.)

Construct the angle **A** of 53°. Along its arms measure off **AB** = 1·55 and **AD** = 1·6. Find the point C_1 by the intersection of two arcs—the first with centre **B** and radius ·73; the second with centre **D** and radius 1·03.

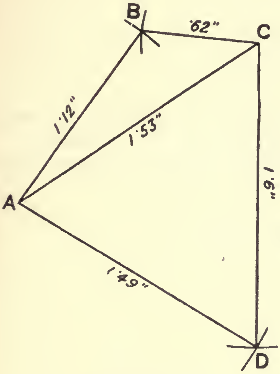

Fig. 41.

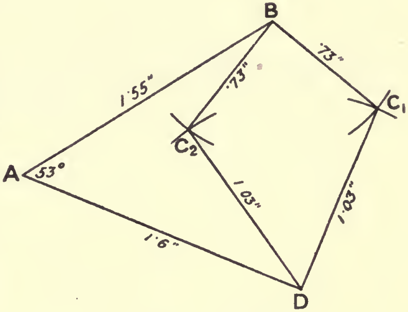

Fig. 42.

Then the quadrilateral ABC_1D satisfies the required conditions.

Note that the two arcs, if produced, will intersect at C_2 as well as at C_1. Thus the "re-entrant" quadrilateral ABC_2D will also satisfy the required condition. A re-entrant quadrilateral can also be constructed in the preceding problem.

Examples.—CVI.

1. In the quadrilateral **ABCD**, **AB** = 1·73 inches, **BC** = 1, **CD** = 1, **DA** = 1·73, **AC** = 2. Construct it, and measure **BD**.

2. In the quadrilateral **PQRS**, **PQ** = 3 inches, **QR** = 2, **RS** = 2, **SP** = 1·73, angle **SPQ** = 90°. Construct it, and measure **PR** and **QS**.

3. In the quadrilateral **HKLM**, **LM** = 1·5 inches, **MH** = 2·5, **HK** = 1·88, angle **LMH** = 120°, **MHK** = 90°. Construct it, and measure **HL** and **KM**.

4. In the quadrilateral **ABCD**, **CD** = 1·2 inches, **DA** = 2, **AB** = 1, **AC** = 2·8, **BD** = 1·73. Construct it, and measure **BC** and the angle **ADC**.

5. In the quadrilateral **PQRS**, **PQ** = 1 inch, **QR** = 1·65, **RS** = 1·25, **SQ** = 2·6, angle **PQR** = 90°. Construct it, and measure **SP** and the angle **SPQ**.

6. In the quadrilateral **ABCD**, **DA** = 2 inches, **AB** = 2, **BC** = 1·41, **CD** = 1·41, angle **BAD** = 60°. Construct the quadrilateral, and measure **AC**.

7. In the re-entrant quadrilateral **HKLM**, **HK** = 3·46 inches, **KL** = 1, **LM** = 1, **MH** = 3, **KM** = 1·73. Construct it, and measure the angle **LMH**.

[Note that one diagonal of a re-entrant quadrilateral (in this case, **KM**) lies without the quadrilateral.]

8. In the quadrilateral ABCD, BAC = 30°, CAD = 30°, ABC = 90°, ACD = 90°, AB = 1·73. Construct it, and measure AD.

9. In the quadrilateral ABCD, BDC = 90°, DBC = 45°, AB = AD = 1·41, BD = 2. Construct it, and measure the diagonal AC.

10. Construct the pentagon PQRST, given that RT = 1·41, PQ = QR = RS = ST = TP = RP = 1. Measure PS and QT.

CHAPTER XVI.

ON AREAS AND VOLUMES.

160. On Areas.—The area of a figure is the measure of its surface.

Theorem I.—Parallelograms on equal bases and between the same parallels are of equal area.

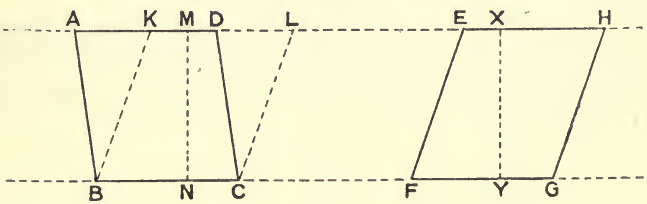

FIG. 43.

This theorem is illustrated by Figs. 43, 44. In Fig. 43 the parallelogram **ABCD** is equal in area to the parallelogram **EFGH**, since their bases **BC** and **FG** are equal, and they are between the same parallels.

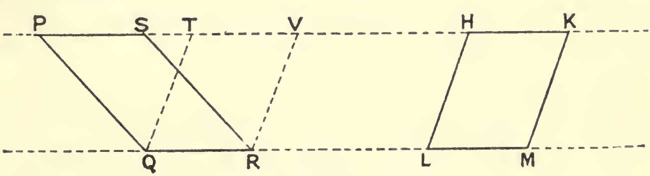

FIG. 44.

In Fig. 44 the parallelogram **PQRS** is equal in area to the parallelogram **HLMK**, for similar reasons.

EXPLANATION.—In Fig. 43 the parallelogram **EFGH** can be placed in the position **KBCL**, because **FG** = **BC**. But we can see that the triangle **ABK** is equal to the triangle **DCL**. It follows that—

the area **ABK** + the area **KBCD** = the area **DCL** + the area **KBCD**
i.e. the area **ABCD** = the area **KBCL**

In Fig. 44 the parallelogram **HLMK** can be placed in the position **TQRV**, because **LM** = **QR**. But we can see that the triangle **PQT** is equal to the triangle **SRV**. It follows that—

the area **PQRV** – the area **PQT** = the area **PQRV** – the area **SRV**
i.e. the area **TQRV** = the area **PQRS**

The **altitude** of a parallelogram is the distance between the base and the opposite side, *measured perpendicularly to the base*. Thus in Fig. 43 **MN** and **XY** are the altitudes of the two parallelograms. (The word "altitude" means "height.")

It is obvious that **MN** = **XY**; thus we see that if two parallelograms are between the same parallels, their altitudes are equal. Theorem I. may, therefore, be expressed as follows :—

Two parallelograms which have equal bases and equal altitudes are of equal areas.

The two parallelograms need not be between the same parallels, for, provided they are of equal altitude, *they could be placed between the same parallels*.

161. Theorem II.—If a triangle and a parallelogram are on equal bases and between the same parallels, the area of the triangle is half the area of the parallelogram.

This is illustrated in Fig 45. **BC** = **FG**, and the area **EFG** is then half the area **ABCD**.

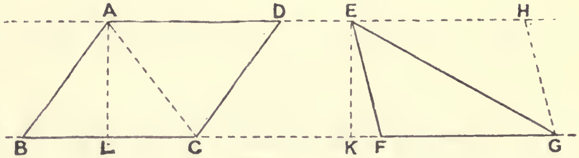

FIG. 45.

EXPLANATION.—If we draw the parallelogram **EFGH**, then its diagonal **EG** divides it into two exactly equal areas (Theorem XIV., Chap. XV.).

Hence the area **EFG** = half the area **EFGH**. Also the area **EFGH** = the area **ABCD** (Theorem I.).

Theorem III.—If two triangles are on equal bases and between the same parallels, they are equal in area.

For example, in Fig. 45 the areas **ABC** and **EFG** are equal.

This follows immediately, as the area of each triangle would be half that of the corresponding parallelogram.

The **altitude** of a triangle is the perpendicular drawn from the vertex of the triangle to the base or the base produced. Thus in Fig. 45 the altitude of the triangle **EFG** is **EK**, which is drawn from the vertex **E**, perpendicular to the base produced. Also the altitude of the triangle **ABC** is **AL**, which is drawn from the vertex perpendicular to the base.

We may express Theorem III. as follows :—

If two triangles have equal bases and the same altitude, they are equal in area.

162. The area of a rectangle is the product of its length and its breadth. This is illustrated in Fig. 46. **(1)**

Suppose that this represents a rectangle whose length **BC** is 5 feet, and whose breadth **AB** is 3 feet. In the figure the rectangle is divided up into squares; and since *each side of every square would represent a foot*, each square represents a **square foot.**

And since we have three rows of squares, each containing five squares, the total number of squares is fifteen.

The argument can be extended to the case where the lengths involve fractions.

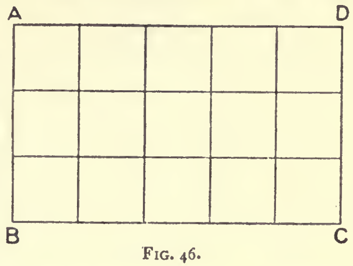

FIG. 46.

The area of a parallelogram is the product of its altitude and its base. **(2)**

This is illustrated in Fig. 47.

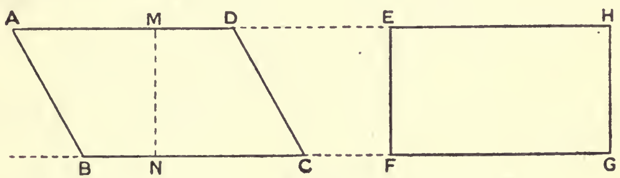

FIG. 47.

Here **BC** = **FG**, and the parallelogram and rectangle are between the same parallels. They are therefore of equal area (Theorem I.).

Thus the area of the parallelogram **ABCD**

= the area of the rectangle **EFGH** = **EF** × **FG** [Formula (1).

= **MN** × **BC** = the product of the altitude and base.

The area of a triangle is half the product of its altitude and base. **(3)**

Thus in Fig. 45 the area of the triangle **ABC** is $\frac{1}{2}$ **AL** × **BC**. This is obvious from Formula (2) of this paragraph, and Theorem XIV. of Chapter XV.

Note that *any side* of a parallelogram or triangle may be regarded as its base.

The area of a right-angled triangle is half the product of the sides which contain the right angle. For if one of these sides be regarded as the base of the triangle, the other will be the altitude.

163. We can now calculate the area of any **rectilineal** figure (*i.e.* any figure bounded entirely by straight lines), by dividing it into triangles.

EXAMPLE.—*Find the area of the figure* ABCDE *in Fig.* 48.

By joining **DA** and **DB**, we divide the area into the three triangles, **ABD**, **BCD**, and **EAD**; draw **DF** perpendicular to **AB**, **DG** perpendicular to

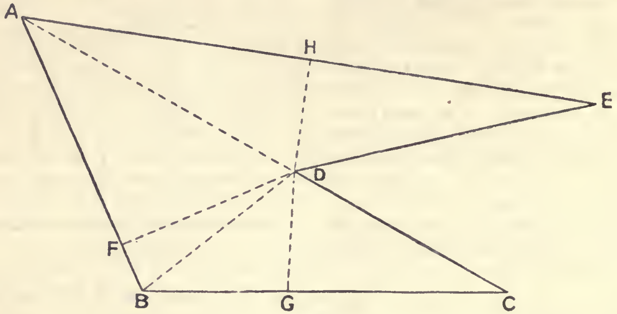

FIG. 48.

BC, and **DH** perpendicular to **EA**; measure **AB**, **BC**, **EA**, **DF**, **DG**, and **DH**.

Then the area of $ABD = \frac{1}{2}AB \times FD = \frac{1}{2} \times 1{\cdot}57 \times {\cdot}99$

,, $BCD = \frac{1}{2}BC \times DG = \frac{1}{2} \times 1{\cdot}92 \times {\cdot}63$

,, $AED = \frac{1}{2}EA \times DH = \frac{1}{2} \times 3{\cdot}18 \times {\cdot}6$

Working out these results, and adding, we obtain, as the area of the figure, 2·34 sq. inches.

164. The Equilateral Triangle.—If a is the length of a side of an equilateral triangle its area is $\frac{a^2\sqrt{3}}{4}$. (4)

EXPLANATION.—Fig. 49 represents an equilateral triangle. The altitude **AD** bisects the base. Thus $AB = a$, $BD = \frac{a}{2}$; also since **ADB** is a right angle—

$$AB^2 = AD^2 + DB^2$$

[Theorem IX., Chap. XV.

$$\therefore AD^2 = AB^2 - DB^2 = a^2 - \left(\frac{a}{2}\right)^2 = \frac{3a^2}{4}$$

hence $AD = \frac{a\sqrt{3}}{2}$

Also the area of the triangle $= \frac{1}{2}AD \times BC$

[Formula (3).

$$= \frac{1}{2} \times \frac{a\sqrt{3}}{2} \times a = \frac{a^2\sqrt{3}}{4}$$

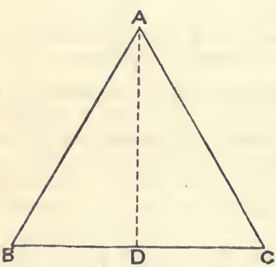

FIG. 49.

165. On Regular Polygons.—Any figure which is bounded by more than four straight lines is called a **polygon**.

There are also a series of special names, according to the number of sides.

A *five*-sided figure is called a			pentagon,
a *six*-sided	,,	,,	hexagon,
a *seven*-sided	,,	,,	heptagon,
an *eight*-sided	,,	,,	an octagon,
a *nine*-sided	,,	,,	a nonagon,
a *ten*-sided	,,	,,	decagon.

A **regular** polygon is one whose sides are all equal and whose angles are all equal.

If we divide the circumference of any circle into any number of equal parts, and join the points of division, in order, by straight lines, we shall obtain a regular polygon.

Thus in Fig. 50 the circumference of the circle has been divided into seven equal parts at A, B, C, etc., and by joining these points there is formed a regular heptagon.

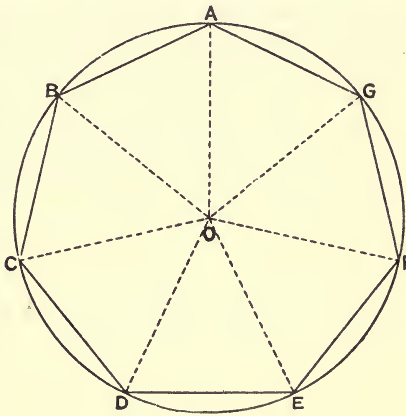

FIG. 50.

EXAMPLE.—*Find the number of degrees contained in each angle of a heptagon.*

In Fig. 50 the seven angles formed at O are together equal to four right angles (by Theorem III., Chap. XV.); moreover, these angles are all equal; hence *each angle at O is four-sevenths of a right angle.*

Again, the three angles of the triangle OBC are together equal to two right angles; and the angle BOC = four-sevenths of a right angle; hence the angles OBC + OCB = ten-sevenths of a right angle; also these two angles are equal to one another, hence *each is equal to five-sevenths of a right angle.*

Thus the angle BCD = twice the angle BCO
= ten-sevenths of a right angle
$= 128\frac{4}{7}°$

EXAMPLES.—CVII.

1. Find the area of a rectangle whose length is 2·38 inches and breadth 1·57 inches.

2. Find the breadth of a rectangle whose area is 7·3414 sq. inches and whose length is 4·14 inches.

3. In a triangle PQR, given that PQ = 2 inches, QR = 3 inches, angle PQR = 50°, construct the triangle, and find its area.

4. Construct a triangle ABC, given that BC = 2·24 inches, angle ABC = 70°, angle BCA = 40°; also find its area.

5. Construct a triangle whose sides are respectively 4, 5, and 6 inches. Measure the length of the perpendicular from each corner to the opposite angle, and find the area of the triangle.

6. Construct a parallelogram whose adjacent sides are 2 and 3 inches respectively, and whose acute angles are each 60°. Determine its area.

7. Construct a parallelogram whose adjacent sides are 2·45 and 3·24 inches respectively, and whose obtuse angles are each 112°. Determine its area.

8. Find the area of an equilateral triangle, each of whose sides measures 3·24 inches.

9. The perimeter of an equilateral triangle is 4·86 inches: find its area. (The perimeter of any figure is the sum of the lengths of its sides.)

10. Find the area of a triangle **ABC**, in which **AB** = 2·38 inches, **BC** = 3·24 inches, angle **ABC** = 90°.

11. If, in Fig. 50, § 165, there were six sides instead of seven, what would be the magnitude of (i.) each angle at **O**, (ii.) the angle **BCD**, (iii.) the angles **OCB, OBC**?

12. Find the magnitude of any angle of a regular octagon, and of a regular pentagon.

13. Construct a regular pentagon within a circle of radius 2 inches, and find its area. (Divide the circumference into five equal arcs by trial.)

14. In a quadrilateral **PQRS**, **PQ** = 2 inches, angle **PQR** = 90°, **QR** = 3 inches, angle **QRS** = 112°, **RS** = 1 inch. Construct the quadrilateral, and find its area.

15. Determine the areas of the figures in Questions 8, 9, 10 of Examples CVI.

166. On Solid Figures.—So far we have only dealt with **plane figures**; that is, figures which lie entirely in one flat surface. We have now to deal with **solid figures**.

Solid figures can only be accurately represented by models. In a

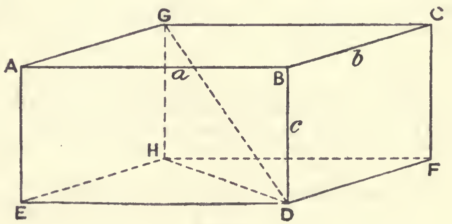

FIG. 51.

drawing of a solid figure, the magnitudes of the lines and angles do not correspond to their real magnitudes in the solid. *The drawing merely represents the appearance of the solid as seen from some definite point of view.* Thus Fig. 51 might be supposed to represent an ordinary box. Now, every angle in an ordinary box is a right angle; yet very few of the angles in this figure are right angles.

This is, perhaps, the most important point for the student to remember with regard to diagrams of solid figures: *angles which are really right angles seldom appear as such in the diagram*; also angles which are really equal seldom appear equal in the diagram.

Definition.—A straight line is said to be perpendicular to a

given plane, if it is perpendicular to every line in the plane which meets it.

Illustrations.—The mast of a ship is perpendicular to its deck; any line in the deck, drawn from the foot of the mast, makes a right angle with the mast.

Also in Fig. 51, regarded as representing a box, the edge BD of the box is perpendicular to its base EDFH; BD makes a right angle with each of the lines DE, DH, and DF, which are all lines in the base of the box.

It should be noticed that a line may be perpendicular to one line in a plane, without being perpendicular to the plane itself. For example, we have just shown that the angle HDB is a right angle; thus the line HD is perpendicular to the line DB, and the latter line lies in the plane BCFD. But HD is certainly not perpendicular to the plane BCFD.

(The line from D perpendicular to this plane BCFD is the line ED.)

Euclid shows, however, in Book XI., that **if a line is perpendicular to *two* intersecting lines in a plane, it is perpendicular to every line in the plane, and therefore to the plane itself.**

167. On Two Planes.—Two straight lines intersect (*i.e.* meet) at a single point. **Two planes intersect in a straight line.**

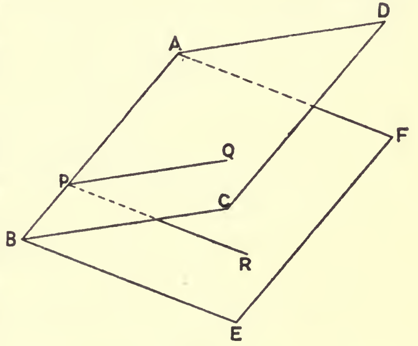

FIG. 52.

For example, in a room, *the plane of the floor intersects the plane of a wall in the straight line which runs along the bottom of that wall;* the plane of the ceiling intersects the plane of a wall in the straight line which runs along the top of that wall. Again, Fig. 52 represents two planes, ABCD and ABEF, which intersect in the line AB.

RULE.—**To measure the angle between two planes, we take any point in their line of intersection, and from this point we draw in each plane a line perpendicular to the line of intersection. The angle between these two lines measures the angle between the planes.**

Thus in Fig. 52 we have taken a point P in AB, the line of intersection of the planes. In the lower plane ABEF, PR is drawn perpendicular to AB (the dotted portion of PR would be hidden by the upper plane); and in the upper plane ABCD, PQ is drawn perpendicular to AB. *Then the angle* QPR *measures the angles between the planes.*

Note that if **BC** and **BE** are both perpendicular to **BA**, the angle **CBE** would also measure the angle between the planes, and would be equal to **QPR**.

The angle between the floor and the wall of a room would be a right angle.

Parallel planes are planes which would never meet, however far they may be extended in any direction.

Illustrations.—The ceiling and floor of a room are parallel planes. Opposite faces of a brick are parallel planes.

168. The Rectangular Block.—This is a figure shaped like a brick or a box (see Fig. 51 or Fig. 53). It has six faces, each of which is a rectangle; only three of them are visible, viz. **ABCG, ABDE, BCFD.** It is completely determined if we know its **length, breadth,** and **depth** (or thickness). The mathematical name for this figure is the **rectangular parallelopipedon.**

The **volume** of a solid body is the *measure of the space which it* occupies.

If the length, breadth, and depth of a rectangular block are a, b, and c respectively—

$$\text{its } \mathbf{volume} = abc \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$
$$\text{its } \mathbf{area} = 2(ab + bc + ca) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

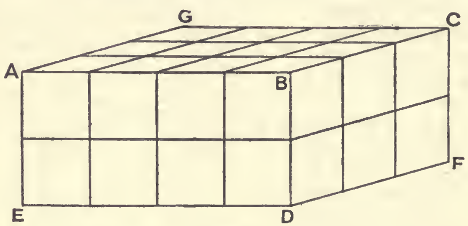

Fig. 53.

Explanation.—Fig. 53 represents a rectangular block, measuring 4 feet long, 3 feet broad, and 2 feet deep. It can be divided up into cubes, as represented in the figure, each cube being **1 cubic foot** in volume, since it *measures* 1 *foot in each direction.*

Since the block is 2 feet deep, there are two layers of cubes; since it is 3 feet broad, each layer contains three rows of cubes; and since it is 4 feet long, each row contains four cubes.

Hence the *total number of cubes in the block is* $4 \times 3 \times 2$, and is therefore given by the product of length, breadth, and depth.

Again, each face is a rectangle, and the area of each rectangle is found by multiplying two adjacent sides.

Thus, using Fig. 51, the area of the top face is **AB** × **BC**, *i.e.* $a \times b$; the bottom face is the same size. The area of the front face is **AB × BD**, *i.e.* $a \times c$; and the same for the back face. The area of each side face is BC × BD, *i.e.* $b \times c$. Thus the total area of the six faces is $2ab + 2bc + 2ca$.

To count the number of edges in a rectangular block, use Fig. 51. We have—

(i.) Four edges of length a, which are parallel to each other, viz. AB, GC, HF, ED.

(ii.) Four edges of length b, which are parallel to each other, viz. BC, AG, EH, DF.

(iii.) Four edges of length c, which are parallel to each other, viz. BD, CF, GH, AE.

Thus we have altogether **twelve edges.**

The three edges HE, HF, HG would not be visible, and are therefore represented in the figure by dotted lines.

The figure has **eight corners**, viz. A, B, C, G, E, D, F, H.

Note that we may express Formula (5) by saying that **the volume is the "product of three adjacent edges."**

The length of the diagonal of a rectangular block is $\sqrt{a^2 + b^2 + c^2}$. **(7)**

EXPLANATION.—In Fig. 51 GD is one of the four diagonals of the rectangular block (the others are AF, BH, CE). Join HD, forming the diagonal of the lowest face of the block.

Now, since HEDF is a rectangle, therefore the angle HED is a right angle. Hence, by Theorem IX., Chap. XV.—

$$HD^2 = HE^2 + ED^2$$

Again, GH is perpendicular to the plane HEDF; and therefore GHD is a right angle. Hence, by Theorem IX., Chap. XV.—

$$GD^2 = GH^2 + HD^2$$

Combining these results, it follows that—

$$GD^2 = GH^2 + HE^2 + ED^2 = a^2 + b^2 + c^2$$

$$\therefore GD = \sqrt{a^2 + b^2 + c^2}$$

169. The Cube.—The **cube** is a rectangular block in which all the edges are equal and all the faces are squares.

We can derive the formulæ for the cube from those of the preceding paragraph, by remembering that b and c are both equal to a.

Hence from Formula (5) we obtain—

$$\textbf{the volume of a cube} = a \times a \times a = a^3 \quad . \quad . \quad \textbf{(8)}$$

From Formula (6) we obtain—

$$\textbf{the area of a cube} = 2(a^2 + a^2 + a^2) = 6a^2 \quad . \quad . \quad \textbf{(9)}$$

From Formula (7) we obtain—

$$\textbf{the diagonal of a cube} = \sqrt{a^2 + a^2 + a^2} = \sqrt{3a^2} = a\sqrt{3} \quad \textbf{(10)}$$

170. The Prism.—The **"right" prism** is a solid figure on a plane rectilineal base; its sides are all rectangles, each of which is perpendicular to the plane of the base; it is bounded above by a figure in all respects equal to its base, and whose plane is parallel to the plane of the base. Figs. 54 and 55 are examples of right prisms. Fig. 54 is on a triangular base **ABC**; Fig. 55 is on a pentagonal base **ABCDE**.

The volume of a right prism is the product of the area of its base by its height. (11)

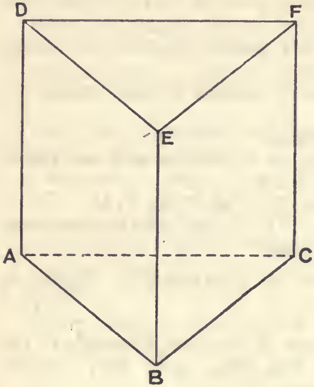

Fig. 54.

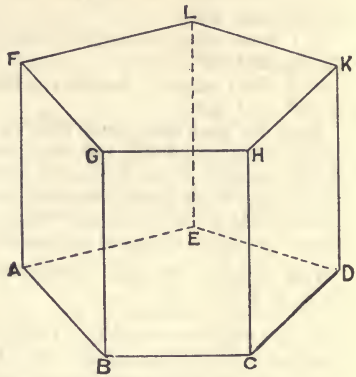

Fig. 55.

Thus the volume of the triangular prism in Fig. 54 is the area of the triangle **ABC** × the length **AD**. Note that **AD** = **BE** = **CF**, and that all these lines are perpendicular to the plane of the triangle **ABC**. Also in Fig. 55, **AF** = **BG** = **CH** = **DK** = **EL**; and all these lines are perpendicular to the plane of the base.

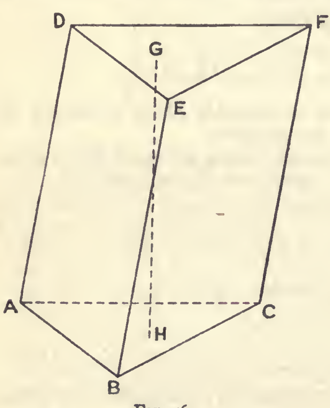

Fig. 56.

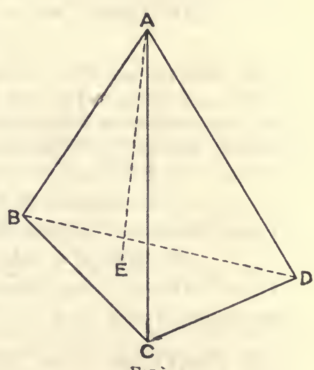

Fig. 57.

In a "**skew**" **prism** the sides are not rectangles, but parallelograms; and their edges are not perpendicular to the plane of the base;

but the upper face is parallel to, and in all respects equal to, the base. Fig. 56 represents a skew triangular prism.

The volume of a "skew" prism is the area of the base multiplied by the perpendicular height of the prism. (12)

Thus in Fig. 56 the perpendicular height of the prism is not measured by DA, as DA is not perpendicular to the plane of the base, but by a line drawn from a point in the upper face perpendicular to the plane of the lower face, such as GH. Thus we multiply the area of the triangle ABC by the length GH.

171. The Pyramid.—A **pyramid** is a solid whose base is a rectilineal figure, and whose sides are triangles having a common vertex at the apex of the solid.

Fig. 57 represents a triangular pyramid whose base is BCD and whose apex is A.

A triangular pyramid is often called a **tetrahedron** (which means a four-faced solid).

Fig. 58 represents a pyramid whose base is a pentagon.

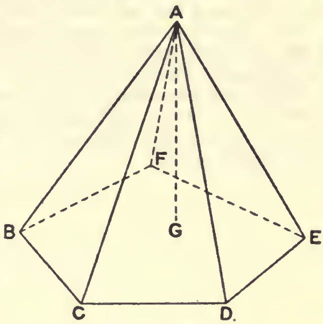

FIG. 58.

The volume of a pyramid is one-third of the product of the area of its base by its perpendicular height. (13)

Thus the volume of the pyramid in Fig. 58 is $\frac{1}{3}$ × the area of BCDEF × the length AG.

In the **regular tetrahedron** each face is an equilateral triangle, and therefore all its six edges are of equal length.

If the length of each edge of a regular tetrahedron is a, its volume is $\dfrac{a^3\sqrt{2}}{12}$. (14)

EXAMPLES.—CVIII.

1. A rectangular block measures 12 feet long, 4 feet wide, and 3 feet deep. Find its volume, its area, and its diagonal. Find also the diagonal of each face of the block.

2. A brick measures 9 inches by $4\frac{1}{2}$ inches by 3 inches. Find its volume. Also find how many bricks form a cubic yard.

3. The dimensions of a room are 20 feet in length, 12 feet in breadth, and 10 feet in height. Find (i.) how many cubic feet of air the room contains; (ii.) how many square feet of paper are required to cover the walls (neglecting doors, windows, etc.); (iii.) the length of a piece of string which will just stretch from one corner of the ceiling to the opposite corner of the floor.

4. A tank measures 5 feet long, 3 feet broad, and 4 feet deep. Find the weight of water which it will hold, given that a cubic foot of water weighs 997 ozs. Also find the number of gallons of water which it will hold, given that 1 gallon of water weighs 10 lbs.

5. A lidless box measures 12 inches long, 10 inches broad, and 7 inches high, these being the external dimensions. If the wood of which the box is composed is 1 inch thick, find how many cubic inches of sand the box will hold; and also the volume of the wood of which the box is made.

6. Find the volume of a right triangular prism whose base is an equilateral triangle of side 2 inches, and whose height is 10 inches.

7. Find the volume of a right triangular prism, given that its height is 12 feet, and that its base is a right-angled triangle whose sides measure 3 feet, 4 feet, and 5 feet respectively.

8. Find the volume and the total area of a regular tetrahedron whose edges are each 3 inches in length.

9. Find the volume of a tetrahedron whose base is an equilateral triangle of perimeter 15 inches, and whose height is 12 inches.

10. Find the volume of a pyramid on a square base, if each side of the base measures 4 feet, and the height is 3 feet.

11. The base of a pyramid is a rhombus in which each side measures 4 feet, and each acute angle measures 60°. Find its volume, if the height of the pyramid is 6 feet.

12. Find the volume of a pyramid whose base is a regular hexagon of perimeter 12 inches, and whose height is 4 inches. (Note that a regular hexagon may be divided into six *equilateral* triangles by lines from its centre to each angular point.)

CHAPTER XVII.

ON THE CIRCLE.

172. The Circle.—The curved line which encloses a circle is called its **circumference.** Any straight line drawn from the **centre** of a circle to the circumference is called a **radius** of the circle. **All the radii of a circle are of equal length.**

Any line which passes through the centre of a circle and is terminated at both ends by the circumference is called a **diameter.** All the diameters of a circle are of equal length; and a diameter is double of a radius.

The ratio of the circumference of a circle to its diameter is 3·14159 . . . correct to six significant figures. (1)

(The proof of this statement involves difficult mathematics, and is beyond the scope of this work.)

It follows that if we know the diameter of a circle, we may find its circumference by multiplying by 3·14159; or that, if we know the circumference, we may find the diameter by dividing by the same quantity.

The ratio 3·14159 . . . occurs in all formulæ connected with the circle and the sphere. It is usual, for convenience' sake, to represent it by the Greek letter π (equivalent to the English *p*).

$\frac{22}{7}$ is often used as an approximate value for this ratio, though it is only correct to three significant figures.

$\frac{355}{113}$ is a very close approximation to the ratio, being correct to six significant figures.

If the radius of a circle is r—

$$\text{its circumference} = 2\pi r \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{and its area} = \pi r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

EXPLANATION.—Formula (2) is obviously the same as (1), since the diameter is double the radius.

A rigorous proof of Formula (3) is somewhat too advanced for the student. But a fair explanation can be given by means of Fig. 59.

If we suppose a very large number of radii drawn from the centre **O** to points **A, B, C,** etc., very close to one another, and at every part of the circumference, these radii will divide the area of the circle into a number of small parts which are approximately triangles. (Nine triangles only are shown in the figure.) The base of each triangle is a portion of the circumference, and the altitude of each triangle is approximately equal to the radius of the circle.

Thus the area of the circle

$$= OAB + OBC + OCD + \text{etc.}$$
$$= \tfrac{1}{2}r \,.\, AB + \tfrac{1}{2}r \,.\, BC + \tfrac{1}{2}r \,.\, CD + \text{etc.}$$
$$= \tfrac{1}{2}r \,.\, (AB + BC + CD + \text{etc.})$$
$$= \tfrac{1}{2}r \times \textbf{the circumference of the circle}$$
$$= \tfrac{1}{2}r \times 2\pi r = \pi r^2$$

N.B.—This is only a rough proof, but the formula is perfectly accurate.

Any portion of the circumference of a circle is called an **arc.** For example, the curve **AEN** in Fig. 59 is an arc. The curve **APN** is also an arc.

The area bounded by two radii and the intervening arc is called a **sector.** Thus the area **OAENO** in Fig. 59 is a sector, being bounded by the radii **OA, ON**, and the arc **AEN**.

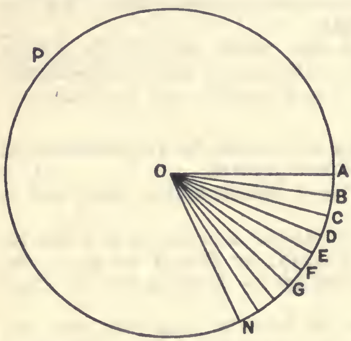

FIG. 59.

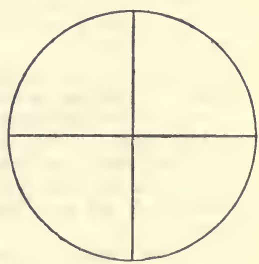

FIG. 60.

The area **OAPNO** is also a sector, being bounded by the radii **OA, ON**, and the arc **APN**.

The area of a sector of a circle is given by half the product of the radius and the arc. (4)

Thus the area of the sector **OAENO** is

$$\tfrac{1}{2}OA \times \text{arc } AEN$$

EXPLANATION.—This follows at once, by considering the area of the sector as the sum of the small triangles represented in the figure.

Again, the area of the sector **OAPNO** is

$$\tfrac{1}{2}OA \times \text{arc } APN$$

Any diameter divides a circle into two equal portions, each of which is called a **semicircle.**

Two radii at right angles form a sector which is called a **quadrant.** Fig. 60 represents a circle divided into four quadrants.

It follows from Formulæ (2) and (3) that

$$\text{the arc of a quadrant is } \frac{\pi r}{2} \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{and the area of a quadrant is } \frac{\pi r^2}{4} \quad . \quad . \quad . \quad . \quad (6)$$

173. The angle subtended by a given line (straight or curved) **at a given point** is the angle formed at that point by *the two straight lines which join the point to the extremities of the given line.*

Thus in Fig. 59 the angle subtended by the arc AE at the centre O is the angle AOE. In any circle, the ratio of any two arcs is equal to the ratio of the angles which the arcs subtend at the centre.

Thus in Fig. 59 arc AF : arc NG = angle AOF : angle NOG.

EXAMPLE (1).—*In a certain circle the arc which subtends an angle of* 130° *at the centre measures* 3·45 *inches : find the length of the arc which subtends an angle of* 16° *at the centre.*

Let the required length be x; then—

$$x : 3{\cdot}45 = 16^\circ : 130$$
$$\text{hence } 130x = 16 \times 3{\cdot}45$$
$$\text{whence } x = {\cdot}425 \text{ inch nearly}$$

It is obvious from Fig. 60 that the *whole circumference of a circle may be regarded as subtending four right angles at the centre,* i.e. 360°.

The following examples are important :—

EXAMPLE (2).—*In a circle whose radius is* 2·5 *inches, find the length of the arc which subtends at the centre an angle of* 48°.

Let x be the length of the arc. The length of the whole circumference $= 2\pi r = 2\pi \times 2{\cdot}5 = 5\pi$ inches; also the whole circumference subtends 360° at the centre, and the required arc subtends 48°. Thus—

$$x : 5\pi = 48^\circ : 360^\circ$$
$$\text{whence } 360x = 240\pi$$
$$\text{i.e. } x = \tfrac{2}{3}\pi = \tfrac{2}{3} \times 3{\cdot}14159$$
$$= 2{\cdot}0944 \text{ inches nearly}$$

EXAMPLE (3).—*Find the angle subtended at the centre of a circle of radius* 5 *inches by an arc of length* 8 *inches.*

Let the angle be x°. The circumference of the circle is $2\pi \times 5 = 10\pi$ inches, and this subtends 360° at the centre. Thus—

$$x^\circ : 360^\circ = 8 : 10\pi$$
$$\text{whence } x = \frac{288}{\pi} = 91{\cdot}67^\circ$$

The **angle of a sector** is the angle between the two radii which bound the sector. The ratio of the area of a sector to the area of the whole circle is equal to the ratio of the angle of the sector to 360°.

EXAMPLE (4).—*Find the area of the sector* OAB *in Fig.* 61.

Measuring the angle AOB, we find it to be 42°. Measuring the radius, we find it to be ·83 inch. Thus the area of the whole circle $= \pi \times (\cdot 83)^2$. Hence, if x is the area of the sector—

$$x : \pi \times (\cdot 83)^2 = 42^\circ : 360^\circ$$

$$\text{whence } x = \frac{\pi \times (\cdot 83)^2 \times 42}{360} = \cdot 2525 \text{ sq. inch}$$

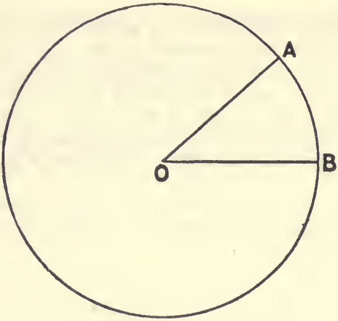

FIG. 61.

EXAMPLES.—CIX.

1. Find the circumference and area of a circle whose radius is 5 inches.
2. Find the circumference and area of a circle whose radius is 2·48 inches.
3. Find the radius and area of a circle whose circumference is 8 inches.
4. Find the area of a circle whose circumference is 1 inch.
5. Find the area of a sector of a circle of radius 2 inches, if the arc of the sector measures 1 inch. Also find the angle of this sector.
6. Find the area of a sector of a circle whose radius is 1 inch, if the arc of the sector is 2 inches. Also find the angle of this sector.
7. Find the area of a sector of a circle whose radius is 2 inches, if the arc of the sector is 6 inches. Also find the angle of this sector.
8. Find the length of the arc of a circle of radius 2 inches which subtends an angle of 80° at the centre.
9. Find what angle is subtended at the centre of a circle of radius 2 inches by an arc of length 2 inches.
10. Find the angle subtended at the centre of a circle of radius a inches by an arc of length a inches.
11. Find the area of a sector whose angle is 44° and whose radius is 2 inches.
12. Find the area of a sector whose angle is ·5° and whose radius is 1000 miles.
13. If the arc of a sector is 10 miles and its radius is 25 miles, find its angle.
14. If the arc of a sector is 1,000,000 miles and its radius 90,000,000, find its angle.
15. If the arc of a sector is 10 inches and its angle 10°, find its radius.

174. The line joining any two points on the circumference of a circle is called a **chord.** Thus in Fig. 50 each side of the heptagon is a chord of the circle.

Any chord divides the area of a circle into two parts, each of which is called a **segment** of the circle.

Thus in Fig. 62 the chord **AB** divides the circle into the two segments **ACB** and **ADB**.

To find the area of the segment **ACB**, we deduct the area of the triangle **OAB** from the area of the sector **OACB**.

To find the area of the segment **ADB**, we add the area of the triangle **OAB** to the area of the segment **OADBO**.

EXAMPLE (1).—*Find the area of each segment in Fig.* 62.

Make the following measurements:—

OA = ·73 inch, **AB** = 1·33, the altitude of the triangle **OAB** = ·29, the angle **AOB** = 134°.

The area of the whole circle = $\pi r^2 = 3{\cdot}14159 \times ({\cdot}73)^2$. Hence the area of the sector **OACB** = $\frac{134}{360} \times 3{\cdot}14159 \times ({\cdot}73)^2 = {\cdot}62316$ sq. inch.

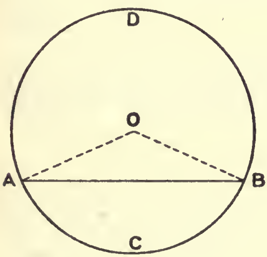

FIG. 62.

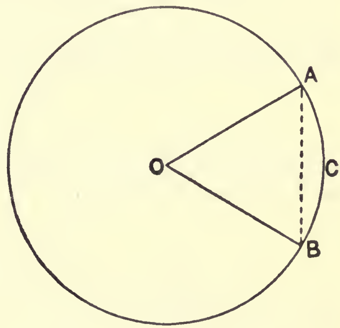

FIG. 63.

Again, the angle subtended at the centre by the arc **ADB** = 360° − 134° = 226°. Hence the area of the sector **OADBO** = $\frac{226}{360} \times 3{\cdot}14159 \times ({\cdot}73)^2$ = 1·051 sq. inch. The area of the triangle **OAB** = $\frac{1}{2} \times 1{\cdot}33 \times {\cdot}29 = {\cdot}1928$ sq. inch. Thus the segment **ACB** = the sector **OACB** − the triangle **OAB** = ·4304 sq. inch. The segment **ADB** = the sector **OADBO** + the triangle **OAB** = 1·2438 sq. inch.

EXAMPLE (2).—*Find the area of the small segment cut off from a circle of radius* 10 *feet by a chord of radius* 10 *feet.*

Use Fig. 63; let **AB** be the chord, then **ACB** is the segment.

Since **OA, OB, AB** are each 10 feet, the triangle **OAB** is equilateral; hence the angle **AOB** = 60°; hence the area of the segment **OACB** is $\frac{60}{360} \times \pi r^2$ = 52·36 sq. feet.

Also the area of the equilateral triangle **OAB** is $\frac{100\sqrt{3}}{4}$ [Formula (4), Chap. XVI.]; *i.e.* 43·3 sq. feet.

∴ the area of the segment = 52·36 − 43·30 = 9·06 sq. feet

175. The following is an approximate formula to find the **length**

of an arc of a circle in terms of the chord of the arc and the chord of half the arc :—

From 8 times the chord of half the arc subtract the chord of the whole arc, and divide the result by 3. (7)

Thus in Fig. 64 the length of the arc ADB is $\frac{8AD - AB}{3}$.

This formula is very accurate indeed if the arc is a small fraction of the circumference. If the arc is one-fourth of the circumference, the error is about ·1 per cent. ; for half the circumference the error is about 1·5 per cent. ; and beyond this the formula does not give satisfactory results.

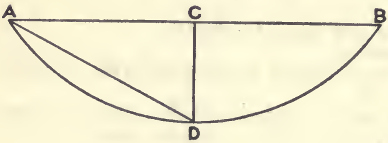

FIG. 64.

A good approximate formula for the area of the smaller segment cut off from a circle by a chord is—

$$\frac{h^3}{2c} + \tfrac{2}{3}ch \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In this formula c represents the length of the chord, and h the "height of the segment," *i.e.* the perpendicular distance of the middle point of the arc from the chord.

EXAMPLES.—CX.

1. Draw a circle of radius 2 inches ; in the circle set off a chord of length 3 inches ; and determine the area of each segment so formed.

2. Find the area of the smaller segment cut off from a circle of diameter 6 inches by a chord of length 3 inches.

3. Find the area of the larger segment cut off from a circle of diameter 4 inches by a chord of length 2 inches.

4. Find the area of the smaller segment cut off from a circle of radius 2 inches, by a chord which subtends a right angle at the centre of the circle.

5. In a circle of radius 1·2 inch place a chord of radius 1·35 inch, and determine the area of the smaller segment so formed.

6. Find the length of an arc of a circle, given that the length of its chord is 2 inches, and the length of the chord of half the arc is 1·2 inches.

7. Find the length of the arc represented in Fig. 64, if AB = 24 inches and CD = 5 inches.

8. A chord of length 8 inches is drawn in a circle of diameter $8\frac{1}{3}$ inches : find the length of the arc of the smaller segment.

9. The length of an arc of a circle is 10·67 inches, and the length of its chord is 8 inches : determine the length of the chord of half this arc ; also the length of the line joining the middle points of the whole arc and its chord.

10. The length of a circular arc is 2·67 inches, and the length of its chord is 2·4 inches : find the distance between the middle points of the arc and chord.

11. Find the area of a segment of height 6 inches on a base of length 12 inches.

12. The chord of a circular arc is 8 inches in length, and the chord of the half-arc is 5 inches in length: find the area included between the whole arc and its chord.

176. Some Important Theorems.—The student should learn the following theorems, and verify them by actual drawing :—

Theorem I.—The straight line drawn from the middle point of a chord, at right angles to the chord, will pass through the centre of the circle. [Euclid, III. 1.]

Draw any two chords in a circle; bisect both chords at right angles. Note that the bisectors always meet at the centre of the circle.

Theorem II.—In the same circle (or in equal circles), equal chords are equidistant from the centre. [Euclid, III. 14.]

Note that the distance of a given point from a given line is measured by the length of the line from the given point perpendicular to the given line. Thus to verify Theorem II., draw a circle with centre **O**, radius 1·5 inch. Draw in this circle two chords, **AB** and **CD**, each of length 1·8 inch. Draw **OE** and **OF** perpendicular to **AB** and **CD** respectively. It will be found that **OE** = **OF** = 1·2 inch.

Theorem III.—In the same circle (or in equal circles), the greater of two chords is nearer to the centre than the less. [Euclid, III. 15.]

Theorem IV.—The angle subtended by an arc of a circle at the centre is double of the angle subtended by the same arc at any point on the circumference. [Euclid, III. 20.]

Draw any circle with centre **O** (see Fig. 65). Draw two radii, **OA, OB**, forming an angle of 70° at **O**. Take any point **P** on the circumference without the arc **AB**. Join **PA, PB**. The angle **APB** will be found to be 35°. Thus the angle subtended by the arc **AB** at **O** is double of the angle subtended by the arc **AB** at **P**.

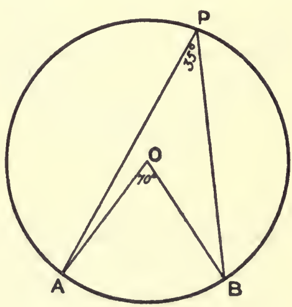

FIG. 65.

Theorem V.—Angles in the same segment are equal to one another. [Euclid, III. 21.]

If **P** is any point on the circumference of a segment whose base is **AB**, the angle **APB** is called an angle "in" the segment.

Draw any chord **AB** of a circle whose centre is **O** (see Fig. 66). Take any points **P** and **Q** on the circumference of the greater segment. Join **AP, BP, AQ, BQ**. On measuring, we shall find that the angles **APB, AQB** are equal; and these are angles "in" the segment **APQB**. By Theorem IV., each of these angles is half the angle **AOB**.

Take any points **R** and **S** on the circumference of the smaller segment. Join **AR, BR, AS, BS**. By measuring, we shall find that the angles **ARB, ASB** are equal; and these are angles "in the" segment **ARSB**.

Note also from this figure that angles in a segment greater than a semicircle are acute; angles in a segment less than a semicircle are obtuse. [Euclid, III. 31.]

Again, we shall find that the angles **APB, ARB** are together equal to 180°. Now, **APBR** is a quadrilateral "inscribed in" the circle; *i.e.* having all its angular points, **A, P, B,** and **R,** on the circumference of the circle. Hence we derive—

Theorem VI.—If a quadrilateral be inscribed in a circle, the sum of either pair of opposite angles is 180°. [Euclid, III. 22.]

Theorem VII.—The angle in a semicircle is a right angle. [Euclid, III. 31.]

Draw a circle and a diameter **AB**. Take a point **P** on the circumference. Join **AP, BP**. The angle **APB** will be found to measure 90°.

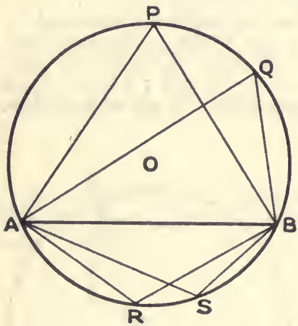

FIG. 66.

Also draw any triangle **HKL,** having a right angle at **L**. Draw the circle on diameter **HK**; *i.e.* the circle whose centre is the middle point of **HK,** and whose radius is $\frac{1}{2}$**HK**. Note that this circle passes through **L**. The angle **HLK** is an angle "in" the semicircle **HLK**.

Theorem VIII.—If two chords **AB** and **CD** intersect at a point **P**, either within or without the circle, then **AP × PB = CP × PD.**

Draw a circle with centre **O** and radius 1·5 inch. Take a point **P** within the circle and exactly 1 inch from **O**. Through **P** draw any two chords, **APB, CPD**. Measuring the lengths, we shall find that **AP × PB = CP × PD** = 1·25. Slight apparent discrepancies will occur, as we can only measure correct to two decimal places. Note that $1·25 = 1·5^2 - 1^2$.

Draw a circle with centre **O** and radius 1·5 inch. Take any point **P** without the circle and exactly 1·9 inch from **O**. Through **P** draw any two lines cutting the circle. If one line cuts the circle at the points **A** and **B**, and the other cuts the circle at the points **C** and **D**, then **AB** and **CD** are two chords of the circle which, when produced, intersect at the point **P**. By measurement we shall find that **AP × PB = CP × PD** = 1·36. Note that $1·36 = 1·9^2 - 1·5^2$.

EXAMPLES.—CXI.

1. In a circle of radius 2 inches place a chord of length 1 inch: measure its distance from the centre.

2. In a circle of radius 3 miles is placed a chord of length 4 miles: calculate its distance from the centre. (From the centre draw a line perpendicular to the chord. This bisects the chord by Theorem I. Join the centre to either extremity of the chord, and use Theorem IX., Chap. XV.)

3. In a circle of radius 2·5 inches draw a chord whose distance from the centre is 1·5 inches, and measure its length.

4. In a circle whose radius is 50 yards is drawn a chord whose distance from the centre is 30 yards: calculate its length.

5. Draw a circle of radius 2 inches. In the circle place a chord of length 2·46 inches. Measure the angle subtended by this chord (i.) at the centre; (ii.) at any point on the circumference of the larger segment; (iii.) at any point on the circumference of the smaller segment.

6. From any point P on the circumference of a circle of radius 1·5 inch, draw chords PQ, PR, including between them an angle of 40°. Measure the length QR, and the angle which QR subtends at the centre.

7. From any point A on the circumference of a circle of radius 2 inches draw two chords, AB, AC, to include between them an angle of 110°. Measure the chord BC, and the angle which BC subtends at the centre.

8. Draw a circle of radius 1·5 inch. Take a point P at a distance ·4 inch from the centre, and through it draw any chord APB. Find the value of AP × PB.

9. Draw a circle of radius 1 inch. Take a point P at a distance 2 inches from the centre, and draw any chord AB which, when produced, will pass through P. Find the value of AP × PB.

177. On Tangents.—A "tangent" to a circle is a line which meets the circumference, but does not cut it, even if produced. A tangent is said to "touch" the circle; and the point where it meets the circumference is called the "point of contact."

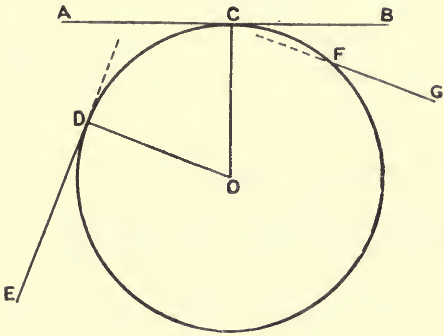

FIG. 67.

Thus in Fig. 67 the line **AB** is a tangent, its point of contact being **C**. The line **DE** is a tangent, for if produced it will not cut the circumference, *i.e.* it will not pass through the circumference. But **FG** is not a tangent, for if produced it cuts the circumference.

Theorem IX.—Any tangent is perpendicular to the radius which passes through its point of contact. [Euclid, III. 16.]

Thus in Fig. 67 **AB** is perpendicular to **OC**, and **DE** is perpendicular to **OD**.

Draw a circle with centre **O**; draw any radius **OA**. Draw **BC** through **A** perpendicular to **OA**. **BC** will be the tangent at **A**.

Problem.—From a given point P without the circumference of a circle, draw the tangents to the circle. (See Fig. 68.)

Join P to O, the centre of the given circle. Describe the circle on diameter PO, cutting the given circle in Q and R. Join PQ, PR; these will be the required tangents.

Join OQ, OR. Note that OQP is an angle in a semicircle, and is therefore a right angle (Theorem VII.). Thus PQ is perpendicular to the radius OQ, and is therefore a tangent.

Theorem X.—If from a point without a circle we draw the two tangents to the circle, the line which joins this point to the centre bisects the angle between the tangents.

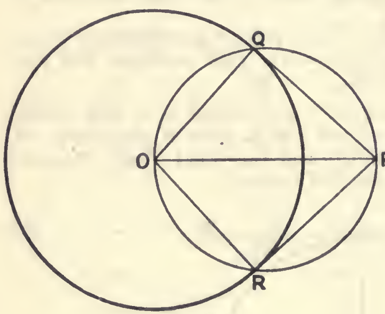

Fig. 68.

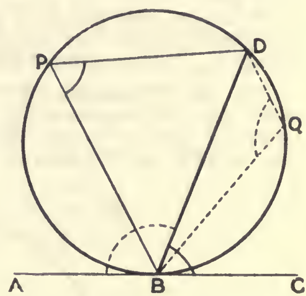

Fig. 69.

Thus in Fig. 68 PO bisects the angle QPR.

Theorem XI.—If from a point P without a circle we draw a tangent PQ touching the circle at Q, and also a "secant" PBA cutting the circle at B and A, then $AP \times PB = PQ^2$.

[A secant is a line which cuts a circle.]

Verify this by actual measurement. Also compare it with Theorem VIII.

Theorem XII.—If ABC is a tangent, and B its point of contact, and if from B we draw a chord BD; also if P is a point on the circumference of the circle, and P and C are on opposite sides of BD; then the angles BPD and DBC are equal.

Also if Q is a point on the circumference of the circle, and Q and A are on opposite sides of BD, then BQD and DBA are equal angles. (See Fig. 69.)

Thus the acute angle between the chord and the tangent is equal to the angle in the greater segment formed by the chord; the obtuse angle between the chord and the tangent is equal to the angle in the lesser segment formed by the chord.

EXAMPLES.—CXII.

1. Take a circle of radius 1 inch. From the centre O draw two radii, **OA, OB**, including an angle of 120°. Draw the tangents at **A** and **B**, and produce them to intersect at **C**. Join **CO**. Measure the angles **ACB, ACO**; also the lengths **CO, CA, CB.**

2. Take a circle of radius 1·5 inches. Take any point A on the circumference. Draw the tangent at **A**, and measure off from it a length **AB** = 1·8 inches. Through **B** draw a line cutting the circle in **C** and **D**. Show that $BC \times BD = 1{\cdot}8^2$.

3. Take a circle of radius 1 inch, and a point **P** at a distance of 1·41 inches from its centre. Draw the tangents from **P** to the circle, and measure their length and the angle between them.

4. In Question 3, place **P** at a distance of 3 inches from the centre, and proceed as before.

5. The two tangents from **P** to a circle of radius 1·5 inches include an angle of 50°. Measure the distance of **P** from the centre.

(The radii to the points of contact will include an angle of 130°.)

CHAPTER XVIII.

THE SPHERE, CYLINDER, CONE, AND ANCHOR RING.

178. The Sphere.—A solid in the shape of a perfectly round ball is called a **sphere.**

Any straight line drawn from the **centre** of a sphere to a point on its surface is called a **radius** ; *all radii of the same sphere are of equal length.* Any straight line passing through the centre of a sphere and terminated at each end by the surface of the sphere is called a **diameter.**

If r is the radius of a sphere—

$$\textbf{the area of its surface} = 4\pi r^2 \quad . \quad . \quad . \quad . \quad (1)$$

$$\textbf{and its volume} = \tfrac{4}{3}\pi r^3 \quad . \quad . \quad . \quad . \quad (2)$$

If a sphere is cut by a plane, the section so formed is a circle ; and the **centre** of the circle is the **foot of the perpendicular** drawn from the centre of the sphere to the cutting plane.

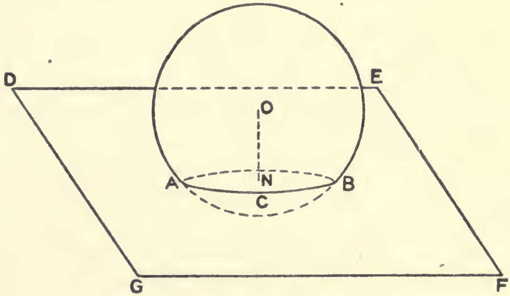

FIG. 70.

Thus in Fig. 70 the plane **DEFG** cuts the sphere in the circle **ABC** ; and the centre of the circle **ABC** is **N**, which is the foot of the perpendicular **ON** drawn from **O** the centre of the sphere to the plane.

If a plane passes through the centre of a sphere, it divides the sphere into two equal halves, which are called **hemispheres.**

The section of the sphere made by such a plane is a circle, whose radius is equal to that of the sphere ; such a circle is called a **great** circle of the sphere. Thus in Fig. 71 AFCE is a great circle, being the intersection of the sphere with a place which passes through O. All circles made by cutting planes which do not pass through the centre have a smaller radius than the sphere; such circles are called **small circles** of the sphere.

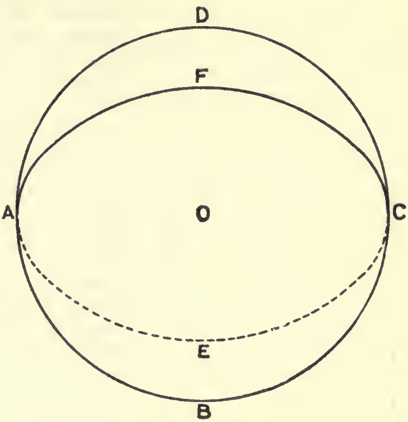

Fig. 71.

The two portions into which a cutting plane divides a sphere are called **segments**. Thus Fig. 72 represents a segment of a sphere. The circle AECF is called the **base** of the segment, and is the section of the sphere made by the cutting plane. D is the centre of the base of the segment; and DB, which is perpendicular to the plane of the base, is called the **height** of the segment.

If the radius of the base of the segment is r, and its height is h—

the volume of the segment $= \frac{1}{6}\pi h(h^2 + 3r^2)$ (3)

and **the area of the curved surface of the segment** $= \pi(r^2 + h^2)$ (4)

Note that these formulæ are correct, *whether the segment is greater or less than a hemisphere.*

The portion of a sphere intercepted between two parallel planes is

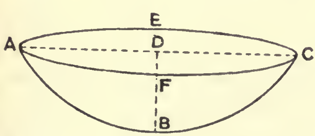

Fig. 72.

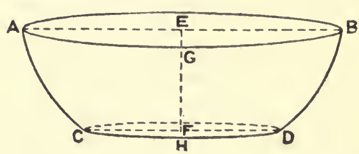

Fig. 73.

called a **zone**. Fig. 73 represents a zone; the circles ABG and CDH are the sections of the sphere made by the two parallel planes.

If AE $= r_1$, CF $= r_2$, EF $= h$, then **the volume of the zone** is given by the formula—

$$\frac{1}{6}\pi h(h^2 + 3r_1^2 + 3r_2^2) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

There is no simple formula for the area of the curved surface of a zone in terms of r_1 and r_2; but if we know the radius of the sphere of which the zone is a portion (which we will call R), the area of the curved surface of the zone is

$$2\pi Rh \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

179. The Cylinder.—The **cylinder** is a solid in the shape of a circular pillar. Fig. 74 represents a cylinder. The base **CDH** and the top **ABG** are equal circles; the line **EF** which joins the centres of these circles, is called the **axis** of the cylinder, and is perpendicular to the plane of each circle. Any section of the cylinder made by a plane parallel to the base is a circle equal to the base.

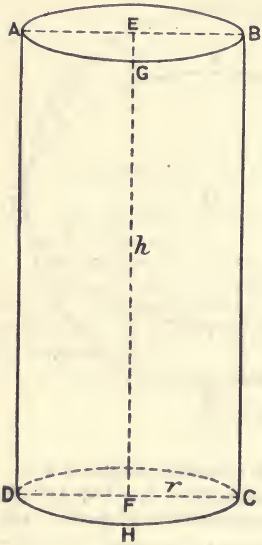

FIG. 74.

The **curved surface** of the cylinder, which stretches from the circumference of the circle **CDH** to the circumference of the circle **ABG**, would (if it were cut down the line **AD** and "peeled off" the cylinder and spread out flat) form a rectangle whose height would be equal to **AD** or **EF**, and whose base would be equal to the circumference of the circle **CDH**. If the radius of the base be r, its circumference would be $2\pi r$ (by Formula (2), Chap. XVII.); hence if the height of the cylinder (*i.e.* **EF**) be h, the area of the curved surface would be $2\pi r \times h$ (by Formula (1), Chap. XVI.). The volume of the cylinder is the area of the base multiplied by the height; and the area of the base is πr^2 (by Formula (3), Chap. XVII.). Thus, if r be the radius of the base and h the height—

the area of the curved surface of the cylinder $= 2\pi rh$. (7)

and **the volume of the cylinder** $= \pi r^2 h$. (8)

To find the total area of the cylinder, we should add the areas of the two circles **ABG** and **DCH** to the area of the curved surface.

N.B.—Strictly we should call this figure a **right circular cylinder,** to distinguish it from a **skew circular cylinder,** in which the axis is not perpendicular to the base, and also from cylinders in which the bases are not circles. But the term "cylinder" may be taken to mean right circular cylinder, unless the contrary is stated.

EXAMPLE.—*Find the radius of a cylinder whose volume is 20 c.c., and whose height is 3cm.*

Using Formula (8).

$$\pi r^2 h = 20$$
$$\text{and since } h = 3$$
$$\text{we have } 3\pi r^2 = 20$$
$$\therefore r^2 = \frac{20}{3\pi}$$
$$\text{whence } r = \sqrt{\frac{20}{3\pi}} = \sqrt{\frac{20}{3 \times 3{\cdot}14159}}$$

Whence, calculating by logarithms, $r = 1{\cdot}456$ cm.

EXAMPLES.—CXIII.

1. Find the area and volume of a sphere whose diameter is 3 inches.

2. Find the area and volume of a sphere whose diameter is 10 cms.

3. Find the volume of a hemisphere whose diameter is 4 cms. Find also its *complete* area.

4. Find the ratio of the area of the curved surface of a hemisphere to the area of its plane surface.

5. A sphere of iron whose diameter is 20 cms. contains a spherical hollow whose diameter is 8 cms. : find the total volume of the iron ; find also its weight, if 1 c.c. of iron weighs 7·5 gms.

6. Find the volume of metal in a spherical shell whose external diameter is 10 cms. and whose thickness is 1 cm. (To do this, subtract the volume of the hollow, which is a sphere of diameter 8 cms., from the volume of the whole sphere.)

7. Find the complete area of a hemispherical bowl of external diameter 8 inches, if the thickness of the metal is 1 inch. (Note that we must calculate (i.) the area of the outer curved surface ; (ii.) the area of the inner curved surface ; (iii.) the area of the rim, which is found by subtracting the area of a circle of diameter 6 inches from the area of a circle of diameter 8 inches.)

8. Find the total area and the volume of a segment of a sphere, if the radius of its base is 4 inches and its height is 1 inch.

9. Find the radius of a sphere, given that the area of its surface is 4 sq. inches.

10. Find the radius of a sphere whose volume is 8 cub. inches.

11. Find the area of a sphere whose volume is 108 c.c.

12. Find the volume of the zone of a sphere, if the radii of its plane surfaces are 3 and 5 inches respectively, and the perpendicular distance between them is 1 inch.

13. The radius of a cylinder is 2 inches, and its height is 10 inches : find the area of its curved surface, its total area, and its volume.

14. The volume of a cylinder is 5 cub. inches, and its radius is ·5 inch : find its height.

15. The area of the curved surface of a cylinder is 10 sq. inches, and its height is 5 inches : find its radius.

16. The volume of a cylinder is 10 cub. inches, and its height is 4 inches : find its radius.

17. A cylindrical pail has an internal diameter of 10 inches : if a gallon of water is poured into it, find the depth of the water. (1 gallon = 277 cub. inches.)

18. How many cubic feet of water are contained in 200 yards of pipes whose internal diameter is 3 inches ?

180. The Cone.—The **right circular cone**, usually called simply "the cone," is a solid on a circular base, tapering to a point at the top. Fig. 75 represents a cone. **BCE** is the circular base ; **D** is the centre of the base ; **DA** is called the **axis** of the cone, and is perpendicular to the plane of the base. The point **A** is called the **apex**. (If **DA** is not perpendicular to the plane of the base, we have a "skew" cone.) Sections of the cone made by planes parallel to the base are circles, diminishing continuously in size as we pass from the base to the apex.

The apex is the same distance from every point in the circumference of the base. This distance, which is usually called the "**slant height**" of the cone, is equal to AB, *i.e.* to $\sqrt{AD^2 + BD^2}$ (by Theorem IX., Chap. XV.).

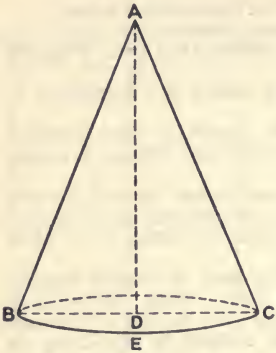

FIG. 75.

The volume of the cone is one-third the product of the area of the base and the height (compare Formula (13), Chap. XVI.); the area of the base is πr^2.

If we cut the curved surface down the line AB, and peeled it off, and spread it flat, we should obtain a sector of a circle, of which A would be the centre and AB the radius; also the length of the arc of the sector would be the circumference of the circle BCE, *i.e.* $2\pi r$. Thus by Formula (4), Chap. XVII., the area of the curved surface $= \frac{1}{2}AB \times 2\pi r$. We thus obtain the following formulæ, if the radius of the base be r, the vertical height h, and the slant height l:—

the area of the curved surface of a cone $= \pi rl = \pi r\sqrt{r^2+h^2}$ **(9)**
and the volume of a cone $= \frac{1}{3}\pi r^2 h$ . . . . **(10)**

The total area would be found by adding the area of the base to that of the curved surface.

181. The **frustum** of a pyramid or cone is the portion intercepted between the base and a plane parallel to the base.

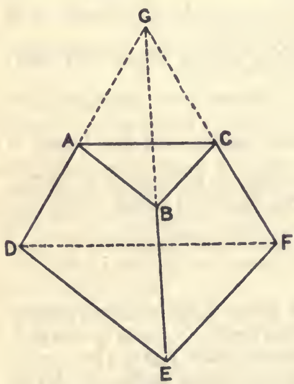

FIG. 76.

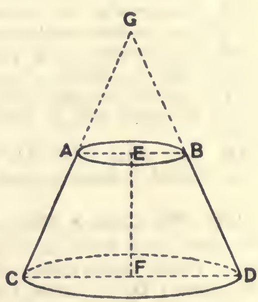

FIG. 77.

Fig. 76 represents a frustum of a triangular pyramid. DEF is the base, and ABC the top of the frustum. The triangle ABC is similar in shape to

DEF, but smaller in size. The **height** of the frustum is the perpendicular distance between the planes **ABC** and **DEF**.

Fig. 77 represents a frustum of a cone ; **EF** is its height.

In either figure **the volume of the frustum**

$$= \frac{1}{3}h(A_1 + \sqrt{A_1A_2} + A_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \textbf{(11)}$$

Where h represents the height, A_1 the area of the base, and A_2 the area of the top.

The area of the **curved surface** of the frustum of a cone is given by the formula—

$$\frac{1}{2}l(c_1 + c_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \textbf{(12)}$$

where l represents the slant height of the frustum (*i.e.* **AC** in Fig. 56), and c_1 and c_2 represent the circumferences of the base and the top of the frustum respectively.

182. The Circular Anchor Ring.—This is represented in Fig. 78. It consists of a solid ring, in which the material has a circular section ; that is to say, a cut through **AB** would give a circle.

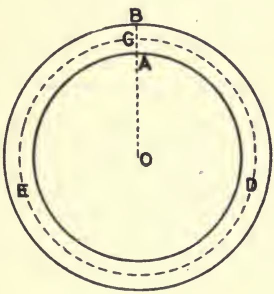

FIG. 78.

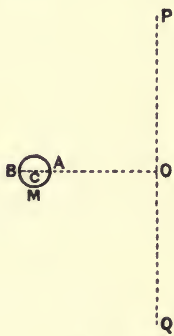

FIG. 79.

OA is called the internal radius, **OB** the external radius, and if **C** is midway between **A** and **B**, then **OC** is called the mean radius. The circle with radius **OC** obviously runs through the middle of the material of the ring ; in fact, every point on this circle is the centre of one of the circular sections of the material.

If Fig. 79 be supposed to revolve about the line **PQ** as axis, then the circle **ABM** would trace out a circular anchor ring ; also this circle would become the circular section of the ring at any point. As before, **OA**, **OB**, and **OC** would be the internal, external, and mean radii.

We may regard this figure as roughly equivalent to a long cylinder bent round till its axis forms a complete circle (the dotted circle **CDE** of Fig. 78), with its top and base in contact (at **AB**, Fig. 78).

Represent the mean radius **OC** by R, and the radius of the circular section **CA** by r. Then the height of the equivalent cylinder = circumference of the circle **CDE** = 2πR ; also the radius of the equivalent cylinder = r. Thus by Formula (7) the curved area of the equivalent cylinder $= 2\pi r(2\pi R) = 4\pi^2 Rr$; and by Formula (8) the volume of equivalent cylinder $= \pi r^2(2\pi R) = 2\pi^2 Rr^2$. Thus—

$$\left.\begin{array}{l}\text{area of circular anchor}\\\text{ring}\end{array}\right\} = \left\{\begin{array}{l}\text{circumference of circular}\\\text{section} \times \text{circumference of mean circle}\end{array}\right\} = 4\pi^2 Rr \quad (13)$$

$$\left.\begin{array}{l}\text{volume of circular anchor}\\\text{ring}\end{array}\right\} = \left\{\begin{array}{l}\text{area of circular section}\\\times \text{ circumference of mean circle}\end{array}\right\} = 4\pi^2 Rr^2 \quad (14)$$

These formulæ are quite accurate, though this method of deriving them is not rigidly correct.

EXAMPLES.—CXIV.

1. Find the volume of a cone, if the area of its base is 10 sq. inches, and its height is 12 inches.

2. Find the volume of a cone, if the radius of the base is 5 inches, and the height is 4 inches.

3. Find the volume of a cone of height 6 cms., if the diameter of the base is 15 cms.

4. What is the height of a cone whose volume is 10 c.c., if the diameter of the base is 4 cms.?

5. What is the radius of the base of a cone whose volume is 10 c.c., and whose height is 10 cms.?

6. Find the total area of a cone whose height is 4 cms., and whose base is a circle of radius 3 cms.

7. Find the ratio of the area of the curved surface to the area of the base in a cone whose height is 12 inches, and whose base is a circle of radius 5 inches.

8. Find the volume of the frustum of a cone, if the radii of the base and top are respectively 4 and 3 inches, and its vertical height is 5 inches.

9. Find the curved area of the frustum of a cone whose slant height is 4 inches, if the radii of its plane faces are 5 and 6 inches respectively.

10. The height of a frustum of a cone is 12 inches, and the radii of its plane faces are 3 and 8 inches respectively: find its volume, and the area of its curved surface.

11. Find the volume of the frustum of a pyramid whose height is 5 inches, if the base is an equilateral triangle of side 3 inches, and its top an equilateral triangle of side 1 inch.

12. Find the volume of a frustum of a pyramid whose height is 4 inches, if the base is a square of side 4 inches, and the top a square of side 2 inches.

13. A cylinder of radius 2 inches and height 6 inches is cut down into the shape of a frustum of a cone on the same base and of the same height, having a top whose diameter is 2 inches: find the volume of the portion which has been cut away.

14. From a solid in the shape of a cylinder of radius 3 cms. and height 8 cms. there is scooped out at each end a hemisphere of the same radius as the cylinder: find the volume of the resulting solid.

15. Find the area and volume of a circular anchor ring whose mean radius is 10 inches, and whose internal radius is 9 inches.

16. Find the area and volume of a circular anchor ring whose external radius is 20 inches, and whose internal radius is 16 inches.

17. What is the diameter of the section of a circular anchor ring whose mean radius is 10 inches, and whose volume is 5 cub. inches?

18. The area of a circular anchor ring is 3 sq. inches, and the diameter of its section is ·5 inch: find its volume.

CHAPTER XIX.

ON SPECIFIC GRAVITY.

183. On Mass and Weight.—In the science of Dynamics it is essential to distinguish between the **mass** and the **weight** of a body; for all other purposes these two terms may be regarded as more or less synonymous.

The distinction is as follows: The **mass** of a body is the "*quantity of matter*" which it contains; the **weight** of a body in the neighbourhood of the earth is the *force with which the earth attracts it;* the weight of a body in the neighbourhood of the moon is the force with which the moon attracts it; and so on.

If we could transfer a body from the earth to the moon, *we should not have altered its mass, but its weight would be very much less,* because the moon, being much smaller than the earth, attracts with far less force.

Again, the weight of a body on the earth's surface changes *very slightly* with its latitude, being *least* at the equator, and *greatest* at the poles. Also if a body is taken up in a balloon, its weight is a *very* little less than at the surface of the earth. If it is taken down a mine its weight *increases* very slightly, but if it could be taken to a sufficiently great depth below the surface, its weight would *decrease,* and at the centre of the earth its weight would be zero.

The definition of mass given above is of no value except *to distinguish mass from weight;* for the term "quantity of matter" is very vague, especially when we wish to compare essentially different kinds of matter, such as water and iron. In reality, we usually compare the masses of two bodies by comparing their weights; hence for ordinary purposes we may regard the two terms as fairly synonymous.

184. On Density.—We measure the **density** of a body by its **mass per unit volume.** Thus, since a gallon of water weighs 10 lbs., we may quote the density of water as 10 lbs. per gallon; we may also quote the density of water as 1 gm. per c.c., etc. The density of silver is about 10,468 ozs. per cub. foot; the density of air at normal temperature and pressure is about 1·27 oz. per cub. foot.

185. On Specific Gravity.—The same property of matter is also measured by the method of **specific gravity.**

The specific gravity of a body is the ratio of its weight to the weight of an equal volume of water.

Thus if we say that the specific gravity of copper is 8·6, we mean that copper is 8·6 times as heavy as water, *comparing equal volumes.*

The specific gravity of water itself is, of course, 1, water being the standard substance.

It follows from the definition given above that—

$$\left.\begin{array}{c}\textbf{the specific gravity}\\ \textbf{of a body}\end{array}\right\} = \frac{\textbf{the weight of the body}}{\textbf{the weight of an equal volume of water}}$$

The statement may also be put into either of these equivalent forms—

the weight of a body = its specific gravity × the weight of an equal volume of water;

$$\frac{\text{the weight of a body}}{\text{its specific gravity}} = \text{the weight of an equal volume of water}$$

Note particularly that, given the specific gravity of a substance, we can always determine the weight of any unit of volume of the substance. For example, the specific gravity of alcohol is ·8; hence, since 1 gallon of water weighs 10 lbs., 1 gallon of alcohol weighs ·8 × 10 lbs. *i.e.* 8 lbs.; also, since 1 cub. foot of water weighs 1000 ozs. (roughly), 1 cub. foot of alcohol weighs ·8 × 1000 ozs., *i.e.* 800 ozs.

186. EXAMPLE (1).—*Find the specific gravity of a piece of metal whose volume is* 27 *cub. inches, and whose weight is* 250 *ozs.*

We first find the weight of 27 cub. inches of water; and then compare this with the weight of the metal.

1728 cub. inches of water weigh 1000 ozs. [§ 44.

∴ 27 ,, ,, ,, $\frac{125}{8}$ ozs.

Thus the specific gravity of the metal

$$= \frac{\text{weight of the metal}}{\text{weight of an equal volume of water}} = 250 \div \frac{125}{8} = 16$$

EXAMPLE (2).—*How many cubic inches are there in* 4 *ozs. of cork?* (*S.g. of cork* = ·25.)

1 cub. inch of water weighs $\frac{1000}{1728}$ oz. (§ 44).

∴ 1 cub. inch of cork weighs ·25 × $\frac{1000}{1728}$ oz. = $\frac{125}{864}$ oz.

$\frac{125}{864}$ oz. of cork measures 1 cub. inch.

∴ 4 ozs. of cork measure 4 ÷ $\frac{125}{864}$ cub. inches = 27·648 cub. inches

EXAMPLE (3).—*Find the weight of a sphere of gold of diameter* 10 *cms.* (*S.g. of gold* = 19.)

The volume of the sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3} \times 3{\cdot}14159 \times 125 = 523{\cdot}5988$ c.c.

1 c.c. of water weighs 1 gm.

∴ 1 c.c. of gold weighs 19 gms.

∴ 523·5988 c.c. of gold weigh 19 × 523·5988 gms. = 9948·377 gms.

EXAMPLE (4).—*Find the radius of a sphere of silver whose weight is* 10 *kilogs.* (*S.g. of silver* = 10·5.)

We will first find the volume of silver which weighs 10 kilogs.

1 c.c. of water weighs 1 gm.

∴ 1 c.c. of silver weighs 10·5 gms.

10·5 gms. of silver measure 1 c.c.

∴ 10 kilogs. *i.e.* 10,000 gms. of silver measure $\frac{10000}{10{\cdot}5}$ c.c. $= \frac{20000}{21}$ c.c.

r is the radius of the sphere, its volume is $\frac{4}{3}\pi r^3$.

$$\text{Hence } \tfrac{4}{3}\pi r^3 = \frac{20000}{21}$$

$$\therefore r^3 = \frac{5000}{7\pi}$$

$$\text{whence } r = 6{\cdot}1034 \text{ cms.}$$

EXAMPLES.—CXV.

(In these questions take the weight of 1 cub. foot of water as 1000 ozs., the weight of 1 gallon of water as 10 lbs., and the weight of 1 cub. centimetre of water as 1 gm.)

1. Find the weight of 144 cub. inches of iron. (S.g. = 7·6.)

2. Find the weight of 5 pints of pure alcohol. (Sg. = ·8.)

3. Find the weight of half a pint of mercury. (S.g. = 13·6.)

4. Find the weight of a cube of cork, whose edge measures 10 cms. (S.g. = ·25.)

5. Find the weight of a sphere of gold of diameter 8 inches. (S.g. = 19.)

6. Find the weight of a rod of silver whose length is 30 cms., and whose section is a circle of diameter 3 cms. (S.g. = 10·5. Use the formula for a cylinder.)

7. Find the weight of a piece of copper wire of thickness $\frac{1}{8}$ inch and of length 20 yards. (S.g. = 8·8.)

8. Find the specific gravity of a piece of metal whose volume is 12 cub. inches, and whose weight is 5 lbs.

9. Find the specific gravity of a piece of metal whose weight is 20 gms., and whose volume is 2·5 c.c.

10. Find the specific gravity of a sphere whose diameter is 20 cms., and whose weight is 1 kilog.

11. Find the specific gravity of a liquid, if five pints of the liquid weigh 8 lbs.

12. Find the specific gravity of a cone of height 20 cms., if the radius of its base is 10 cms., and its weight is 2 kilogs.

13. Find the specific gravity of a frustum of a cone of height 6 cms., if the radii of its plane surfaces are respectively 4 and 6 cms., and its weight is ·5 kilog.

14. Find the volume of 20 lbs. of lead. (S.g. = 11·4.)

15. Find the volume in pints of 5 lbs. of pure alcohol. (S.g. = ·8.)

16. Find the volume of 1 kilog. of naphtha. (S.g. = ·85.)

17. Find the radius of a sphere of silver whose weight is 100 gms. (S.g. = 10·5.)

18. What length of copper wire of thickness ·2 cm. will weigh 1 kilog.?

187. On the Practical Determination of Volumes and Densities.—Fig. 80 shows a method for determining the volume of a small irregularly shaped solid.

A is a vessel to hold water, which is fed by a tube entering at **B**. The flow of water is controlled by a tap. The spout of the vessel, **C**, is *above* the level of **B**. Water is admitted into **A** through the tube till it is *exactly* at the level of the spout. This exact level is best obtained by regulating the tap till the water is dripping slowly in drops from **C**; if the tap be then turned off, the dripping will cease, and the exact level of water is secured. An empty cup D *of known weight* is then placed below the spout, and the solid K is *very gently* lowered into the water by means of a fine thread. During this process water flows from C into D; and if the solid is lowered with sufficient care the level of the water remains exactly at the level of the spout. In this case **the water which has flowed into D will have exactly the same volume as K.** (If the solid is lowered more quickly, there will be a little disturbance caused in the water in A, and too much water will flow out at C.)

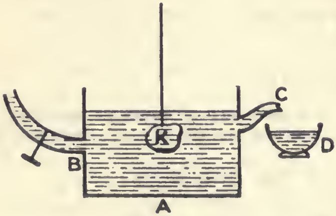

FIG. 80.

If we now weigh the cup D with the contained water, and subtract the weight of the cup itself, we shall obtain the exact weight of the water in D. From this we can calculate the volume of the water in D by the methods of the preceding paragraphs. This result gives the volume of K.

For example, if the empty cup weighs 27·52 gms., and if with the contained water it weighs 41·24 gms., then the weight of the water in D = 41·24 − 27·52 = 13·72 gms. Hence the volume of the water in D = 13·72 c.c., which is equal to the volume of K.

If the body floats in water, it must be fastened to the end of a fine wire and very gently pushed below the surface.

Note that by § 185 we can find the specific gravity of the body by dividing the weight of the body by the weight of the water in D.

188. Another Method.—If a body be suspended by a thread and immersed in water, it will apparently weigh less than in air. Fig. 81 shows the method of weighing a body in water. One scale-pan is on a higher level than the other, and the body is suspended by a fine thread from a hook under this scale-pan, and hangs immersed in the water below.

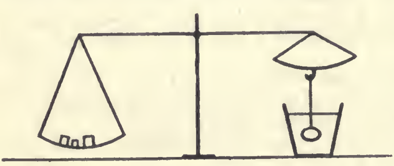

FIG. 81.

It is a well-known principle in Hydrostatics that **the difference in the weights of a body in air and in water is equal to the weight of an equal volume of water.**

EXAMPLE.—*A piece of metal weighs* 2·31 *ozs. in air and* 2·09 *ozs. in water: find its volume, and also its specific gravity.*

The weight of an equal volume of water is 2·31 − 2·09 ozs., *i.e.* ·22 oz.
But 1000 ozs. is the weight of 1728 cub. inches of water.

∴ ·22 oz. is the weight of ·22 × 1728 ÷ 1000 cub. inches of water.

Thus the volume of the body = ·22 × 1728 ÷ 1000 cub. inches = ·38 cub. inch.

By § 185 the specific gravity of the body = the weight of the body ÷ the weight of an equal volume of water = 2·31 ÷ ·22 = 10·5.

EXAMPLES.—CXVI.

1. If, in the method of § 187, the weight of water "displaced" by the body (*i.e.* of the water which flows over into the cup) is 2·5 ozs., determine the volume of the body.

2. If the weight of a piece of metal is 13·5 ozs., and its specific gravity is 10·5, what weight of water will be displaced by the metal in the experiment of § 187?

3. What is the specific gravity of a body whose weight is 50 gms., if in the experiment of § 187 it displaces 4 gms. of water?

4. Determine the volume and specific gravity of a solid body, whose weight in air is 100 gms., and whose weight in water is 94·8 gms.

5. Determine the weight in water of a body whose weight in air is 17 ozs., and whose specific gravity is 8.

6. If the specific gravity of a body is S, find the ratio of the weight of the body in air to its weight in water. Hence find the weight in air of a body whose specific gravity is 8, and whose weight in water is 105 gms.

CHAPTER XX.

ON THE PRACTICAL DETERMINATION OF AREAS.

189.—To reduce a quadrilateral to a triangle of equal area.

RULE.—**To reduce a quadrilateral X O I II to a triangle, one side of the triangle to be X II, and the other to lie along XO. Produce XO; join O II; draw I 1, parallel to O II, meeting XO**

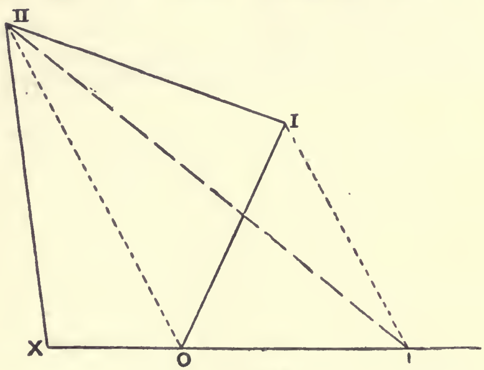

FIG. 82.

(or XO produced) in 1; join II 1. X 1 II is the required triangle. (See Figs. 82 and 83.)

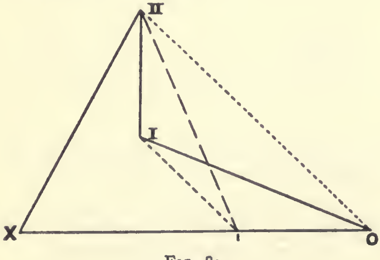

FIG. 83.

EXPLANATION.—In Fig. 82 the triangle **O1 II** is equal in area to the triangle **OI II**—for they are on the same base **O II**, and between the same parallels **O II** and **I 1** (Theorem III., Chap. XVI.).

Add to each the triangle **XO II**; then—

XO II + O 1 II = XO II + OI II

i.e. the triangle **X 1 II** = the quadrilateral **XO I II**

Again, in Fig. 83 (which represents a quadrilateral with a "**re-entrant angle,**" **OI II**), as before, the triangle **O1 II** is equal in area to the triangle **OI II.**

Subtract each from the triangle XO II ; then—

XO II – O 1 II = XO II – OI II,

i.e. the triangle X 1 II = the quadrilateral XO I II

190.—To reduce a pentagon to a triangle of equal area.

RULE.—**To reduce a pentagon XOI II III** (Fig. 84) **to a triangle, one side of the triangle to be X III, and the other side to lie along**

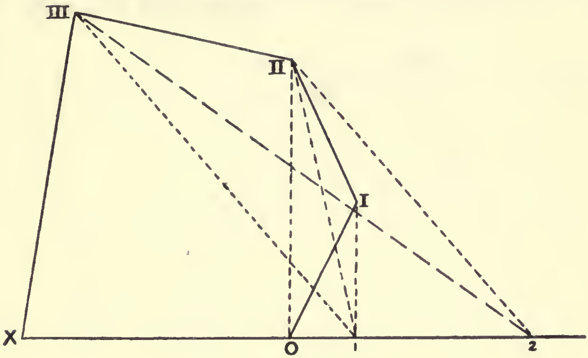

FIG. 84.

XO. Produce XO; draw I 1 parallel to O II; draw II 2 parallel to 1 III; then X 2 III is the required triangle.

EXPLANATION.—By Theorem III., Chap. XVI.—

O 1 II = O I II ; add to each XO II III—

then XO II III + O 1 II = XO II III + O I II

i.e. quadrilateral X 1 II III = pentagon XO I II III

Again 1 2 III = 1 II III ; add to each X 1 III

then X 1 III + 1 2 III = X 1 III + 1 II III

i.e. triangle X 2 III = quadrilateral X 1 II III

hence the triangle X 2 III = the pentagon XO I II III

The rule will also hold if the pentagon has re-entrant angles.

It is obvious, from the method of using parallel rulers or marquois scales, that *we need not actually draw any of the dotted lines except* 2 III, but merely place the ruler in position for drawing the parallels, and *mark off the points* 1 *and* 2.

191. By extending this method **we may reduce any rectilineal figure** to a triangle. Thus Figs. 85 and 86 represent the reduction of two irregular hexagons, one of them with a re-entrant portion. The re-entrant portion makes no difference to the rule, which is quite mechanical and easily remembered if the points are numbered as in the figures. (With a re-entrant portion the figures **0, 1, 2, 3,** do not come in order along the line, and it may or may not be necessary to produce XO.) Notice the regular sequence of the numbers in the rule.

RULE.—Draw I 1 parallel to O II ;
„ II 2 „ „ 1 III ;
„ III 3 „ „ 2 IV.

Then X 3 IV is the required triangle.

This is the easiest method of determining the area of any

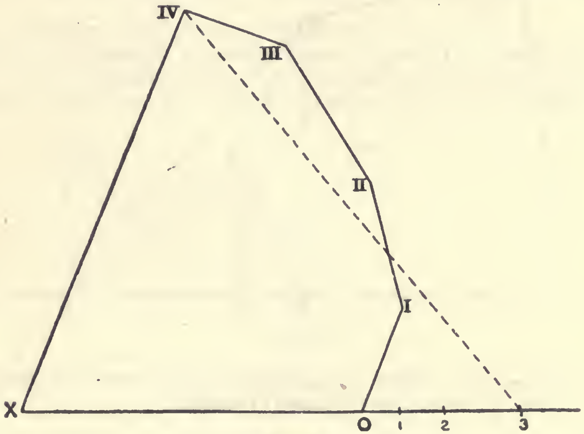

FIG. 85.

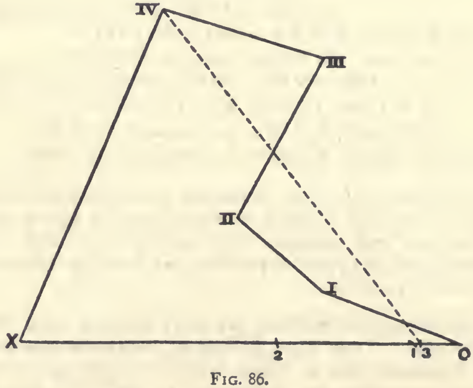

FIG. 86.

rectilineal figure except parallelograms, as the area of the triangle is half the product of its base and altitude.

EXAMPLES.—CXVII.

1. Construct the quadrilateral **ABCD** from the following data, and find its area: **AB** = 2·2 inches, angle **ABC** = 80°, **BC** = 1·5 inch, angle **BCD** = 110°, **CD** = 1 inch.

2. Construct the quadrilateral **PQRS** from the following data, and find its area: **PQ** = 2·2 inches, angle **PQR** = 80°, **QR** = 2 inches, angle **QRS** = 20°, **RS** = 1·6 inches.

3. Construct a regular pentagon in a circle of radius 2 inches, and determine its area.

4. In a circle of radius 3 inches construct a regular polygon of twelve sides; take five consecutive angular points of this polygon, numbering them **O, I, II, III, IV**; join **O** and **IV** to the centre **X**. Reduce the figure **XOI II III IV** to a triangle, and determine its area.

5. In a circle of radius 2 inches construct a regular octagon, and determine its area.

6. Six lines are drawn from a point **O**, viz. **OA, OB, OC, OD, OE, OF**; the angle between each pair of lines is 60°; **OA** = 4 inches, **OB** = **OF** = 3 inches, **OC** = **OE** = 2 inches, **OD** = 1 inch. Determine the area of the rectilinear figure **ABCDEF.**

192. To determine the area between a small arc and its chord.—The following rule gives correct results, provided the arc is *fairly regular in shape, is comparatively "flat," and is curved in one direction only.* Thus the rule will apply in Fig. 87, but not in Fig. 88, as the

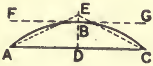

FIG. 87.

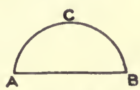

FIG. 88.

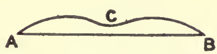

FIG. 89.

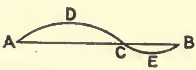

FIG. 90.

arc recedes too far from the line; nor in Fig. 89, as the arc is curving inwards at **C**; in Fig. 90 it may be applied separately to the two portions **ADC, CEB.**

RULE.—**Construct a triangle on the chord as base, having an altitude equal to four-thirds of the distance between the chord and the parallel tangent.**

Thus in Fig. 87 parallel to the chord **AC** we draw the tangent **FBG.** From the point of contact **B** we draw **BD** perpendicular to **AC.** Then **BD** *is the distance between the chord and the parallel tangent.* We then construct the triangle **ACE,** whose altitude **ED** is four-thirds of **BD.** Determining the area of this triangle, we have, approximately, the area enclosed between the arc **ABC** and the chord **AC.**

With regard to the degree of accuracy of this rule, it may be mentioned that if applied to the arc which forms one-third of the circumference of a circle the error is about 6 per cent.; if applied to one-sixth of the circumference, the error is only about 1·5 per cent. The rule is exact for any portion of a *parabola*, from which curve the rule is derived.

By means of this rule we can find approximately the area of any curve. For example, in Fig. 91 it is required to find the area of the ellipse **ACE**. The points **A, B, C, D, E, F** divide the curve into a sufficient number of fairly "flat" arcs. By applying the above rule the area included between the arc **AB** and the chord **AB** is replaced by the triangle **AHB**; the area included between the arc **AF** and the chord **AF** is replaced by the triangle **AKF**; and so on. Thus the area included by the curve is equal to the rectilineal area **F K A H B X C I D III E V.** Joining **XV**, we cut this figure into two portions, and reduce each

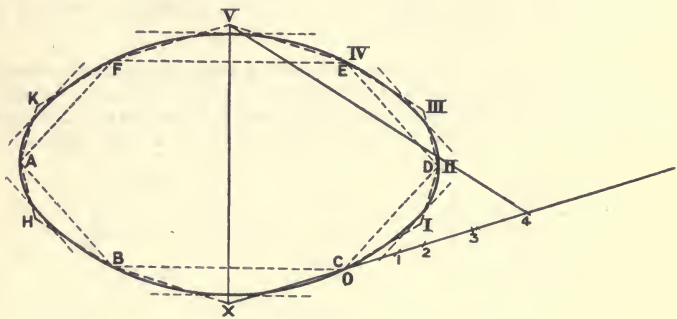

FIG. 91.

portion to an equivalent triangle. In the figure the angular points of the polygon have been renamed **X, O, I, II, III, IV, V**, to facilitate the application of the rule of § 184; following out this rule, we obtain **X4V** as the equivalent triangle. The left-hand portion of the figure must next be reduced to a triangle, and the area is then easily determined. In this particular figure the two portions are exactly equal, and the required area is therefore double the triangle **X4V**.

The student must note that the most convenient type of figure for the rule of § 191 is one in which *the angle marked* **X** *lies between* 50° *and* 100° *roughly, and in which the side* **XV** *is the longest in the polygon.* In the original polygon **F K A H B X C I D III E V**, all the angles were large and all the sides small; but by drawing the diagonal **XV** we obtain two polygons of the required type, which are easily reduced.

EXAMPLES.—CXVIII.

Determine by the method of this paragraph the areas of the following figures :—

1. A quadrant of a circle of radius 2 inches.
2. A semicircle of radius 1·85 inches.
3. A sector of a circle of radius 2·5 inches, whose angle is 120°.
4. A sector of a circle of radius 3 inches, whose angle is 57°.
5. The smaller segment of a circle of radius 2 inches, which is formed by a chord of length 3 inches.

193. It is frequently required in practical work to find **the area included between three lines at right angles, and a curve ;** such as the area of **ABCDEF** in Fig. 92. In this case a slight modification

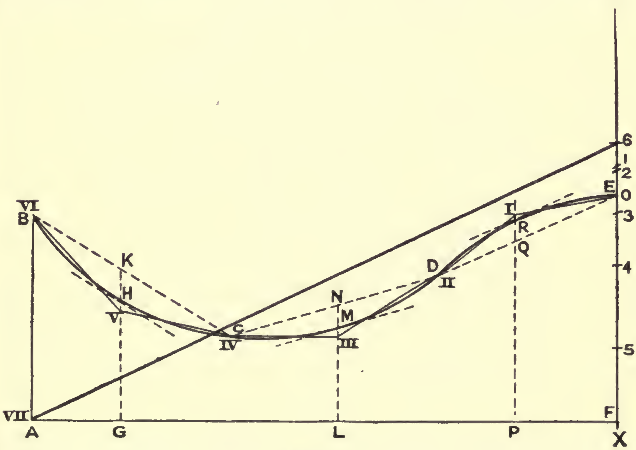

FIG. 92.

of the rule of § 192 is a little more convenient. As before, tangents are drawn parallel to the chords **BC, CD, DE,** touching the curve at the points **H, M,** and **R** respectively. Through these points of contact, **H, M,** and **R,** we draw "**ordinates**" (*i.e.* lines perpendicular to the base-line **AF**) ; *and we use the portions of these ordinates intercepted between the arc and the chord,* instead of the perpendicular distance between the tangent and the chord. Thus we set off **KV** = $\frac{4}{3}$**KH, N III** = $\frac{4}{3}$**NM,** and **QI** = $\frac{4}{3}$**QR**. Then the area required is equivalent to the rectilineal area **A B V C III D I E F,** which is easily reduced to the triangle **X 6 VII.**

Note again, that in renaming the points of this figure, the angle named **X** is a right angle, and the side **X VII** is the longest side of the figure.

This method is also available if either or both of the extreme ordinates are of zero length, that is, if B coincides with A, or E with F. If both the extreme ordinates are zero, we have the area between a curve and its chord.

194. Another method of finding an area of this type is by drawing a number of ordinates at equal distances apart; measure these ordinates; obtain the average of their lengths; and multiply by the length of the base-line. (See Fig. 93.) This is expressed by the

RULE.—**Multiply the base-line by the average length of the ordinates.**

This rule is not very accurate; it gives almost exactly the area included by the rectilinear figure formed by joining the tops of the ordinates.

In Fig. 93 the lengths of the ordinates in inches, beginning with AB and finishing with CD, are respectively ·71, ·98, 1·20, 1·36, 1·49,

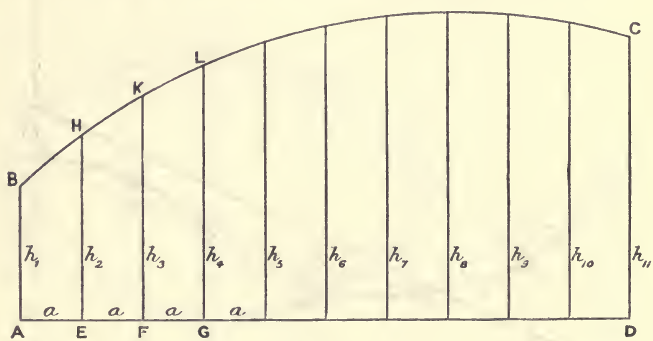

FIG. 93.

1·56, 1·61, 1·63, 1·62, 1·58, 1·50. Adding these lengths and dividing by 11, we find the average length to be 1·385. Multiplying this by 3·26, the length of AD, we obtain the result 4·515 sq. inches.

The following rule gives a much closer approximation:—

SIMPSON'S RULE.—**Divide the base into an *even number* of equal parts, and measure the ordinates erected at each point of division. Add together the first and last ordinates, twice the sum of the other odd ordinates, and four times the sum of the even ordinates; multiply the result by one-third the distance between consecutive ordinates.**

Thus in Fig. 93, if h_1, h_2, h_3, etc., be the lengths of the successive ordinates, and a the distance between each pair of ordinates, the formula for the area becomes—

$$\tfrac{1}{3}a\{h_1 + h_{11} + 2(h_3 + h_5 + h_7 + h_9) + 4(h_2 + h_4 + h_6 + h_8 + h_{10})\};$$

substituting the values of the ordinates given above, and noticing that

$a = \frac{1}{10}AD = \cdot 326$, we obtain the value 4·617 sq. inches. This is correct to within ·2 per cent.

NOTE 1.—If the base is divided into an even number of parts, there must be an odd number of ordinates.

NOTE 2.—If the curve is of fairly regular shape, it will be sufficient to divide the base of the area into six or eight equal parts, but if the curve is irregular or "wavy," a larger number of divisions will be required.

195. Either of the methods of the preceding paragraph may be applied to a closed curve. Thus in Fig. 94 draw two parallel tangents **AB** and **CD.** Draw the line **EF** perpendicular to **AB**, and divide it

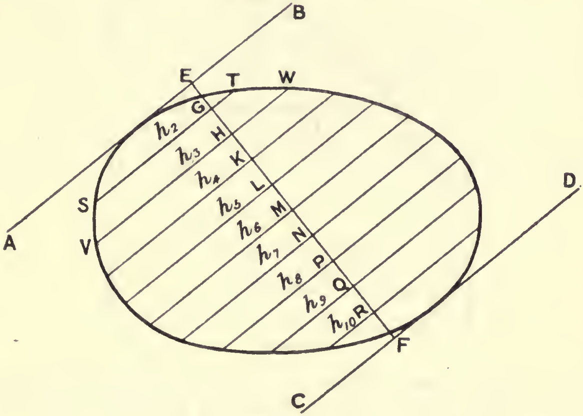

FIG. 94.

into an even number of parts at the points **G, H, K,** etc. Through these points draw **ST, VW,** etc., parallel to **AB.** These lines measure the "**breadths**" of the curve at **G, H, K,** etc., and are used in the calculation in place of the ordinates of the preceding paragraph.

Note that the breadth of the curve at either tangent AB or CD is zero.

We may then apply the formula for Simpson's Rule given in the preceding paragraph, remembering that $h_1 = 0$, $h_{11} = 0$, and $a = EG$.

196. On the determination of Areas by Squared Paper.—Another method of determining the area of a figure is to draw the figure on paper ruled in small squares of a known area; the number of *whole squares* included within the area can be counted, and the value of the *portions of squares* included within the area can be estimated with fair accuracy by the following

RULE.—**Reckon portions of a square greater than a half, as whole squares; reckon portions of a square less than a half,**

as nothing; portions which appear to be exact halves should be reckoned as such.

As an example, let us find the area of the quadrant of a circle in Fig. 95. Suppose that each side of every square formed by the ruled lines measures half a centimetre; hence the area of each square is a quarter of a square centimetre. For convenience of counting, we divide the area by the dotted lines into convenient groups of squares. The area marked A contains nine rows of four squares each; B contains four rows of five squares each; C contains eleven squares; thus the total number of whole squares is $36 + 20 + 11 = 67$. The

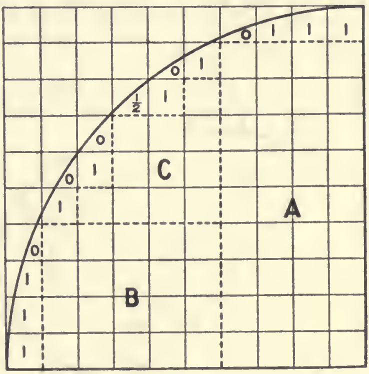

FIG. 95.

portions of squares are marked 1, 0, or $\frac{1}{2}$ in the figure, according as they are greater than, less than, or equal to half a square; then, by the above rule, we reckon these as equivalent to $10\frac{1}{2}$ squares. Thus we count the figure as equivalent to $67 + 10\frac{1}{2}$, which is $77\frac{1}{2}$ squares; this gives the area as $77\frac{1}{2} \times \frac{1}{4}$ sq. cm., *i.e.* 19·4 sq. cms.

We may test this result by applying the formula for the area of a quadrant of a circle whose radius is 5 cms. This gives $\frac{1}{4}\pi r^2 = \frac{1}{4} \times 3{\cdot}1416 \times 25 = 19{\cdot}6$ sq. cms.

It is obvious that this method is not so accurate as the method of replacing the curve by a rectilinear figure; but it is decidedly less troublesome; and, provided the figure includes a large number of squares, the error should be less than the area of one square.

EXAMPLES.—CXIX.

1. Draw a circle of radius 2 inches; draw any diameter, and divide it into three equal parts; through the points of division draw chords perpendicular to the diameter: find the area of each portion into which the circle is divided, both by Simpson's Rule and by using squared paper.*

* Note that to find the exact length of the side of a square on squared paper, it is best to measure the length of ten sides, and divide it by 10.

2. Draw a circle of radius 2 inches; draw any diameter; at distances of half an inch from one end and 1 inch from the other end of this diameter draw chords perpendicular to the diameter. By Simpson's Rule, and also by squared paper, determine the area of that portion of the circle which is included between these two chords.

3. Find, both by Simpson's Rule and by squared paper, the area of the smaller segment cut off from a circle of radius 2 inches by a chord of length 3·46 inches.

4. Use Simpson's Rule to find the area of a circle of radius 2 inches.

5. Construct a square ABCD of side 3 inches, whose diagonals intersect at O. With centre O describe the semicircle DAB; with centre C and radius CD describe an arc DB. Determine, by Simpson's Rule, the area between these two curves.

197. The Planimeter.—Any instrument for measuring an area is called a **planimeter**. All planimeters in general use are on the principle of Amsler's planimeter, a diagrammatic representation of which is given in Fig. 96. The area PQR which is to be measured, and which may be of any shape, is spread on a smooth, flat surface, such as a drawing-board, and pinned down.

The planimeter consists of two bars AB, BD, freely hinged at B. At the end A is a pivot, which is fixed to the drawing-board; a

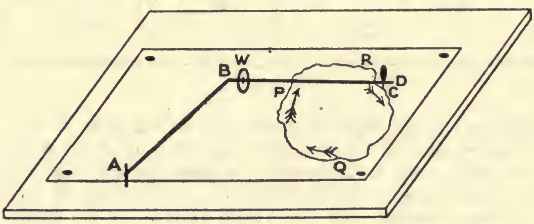

FIG. 96.

wheel W revolves round an axis which coincides with the arm BD; the wheel rests on the surface of the paper. At D is a tracing-point C. There is also some mechanical apparatus, not represented in the figure, for registering the area, which depends on the exact number of revolutions and fraction of a revolution made by the wheel W.

It is obvious that if the bar BD were disconnected and moved parallel to itself, the wheel W would roll on the paper *without slipping*; also that if the bar BD were moved in the direction of its length, the wheel would slip over the paper *without rolling*. The actual motion of the wheel, as the tracing-point C is moved, is generally a combination of slipping and rolling.

In order to find the area PQR, we first set the wheel W so that it registers zero; with the tracing-point C at some marked point on the boundary-line PQR, we then move C in the "clockwise" direction round the line PQR till we come back to the starting-point; *i.e.* we move C completely round the line PQR in the direction indicated by the arrows; the reading of the wheel W then gives the area of PQR.

The mathematical theory of this instrument is too advanced for this book; the important points in the use of it are (i.) the "clockwise" direction, *i.e.* moving the tracer round the area in the same direction as the hands of a clock travel round its face; (ii.) that the reading of the wheel has no practical meaning, *unless the tracer* C *has returned to its starting-point.*

198. The Vernier.—As it is probable that the exact reading of the wheel is shown by a vernier, it will be well to explain here the principle of the vernier, which is applied to many instruments.

In Fig. 97 PQ represents a portion of a graduated scale; RS is a slide which runs along it. It is required to find the exact reading on the scale PQ indicated by the arrow on RS, which is marked A on the figure. The reading is obviously a fraction more than 48, since the graduation marked D reads 48. The denominator of this fraction is

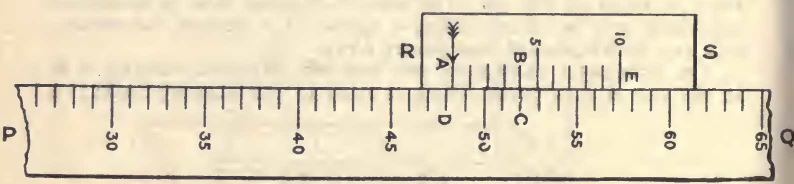

FIG. 97.

always indicated by the highest graduation on the slide, and in this case is 10; the numerator of the fraction is always indicated by that graduation on the slide which coincides (or most closely coincides) with a graduation on the scale; in this case the graduation B on the slide coincides almost exactly with the graduation C on the scale; but the reading of B on the slide is 4, which is therefore the numerator of the fraction. Thus the reading indicated is $48\frac{4}{10}$.

EXPLANATION.—It will be found that the distance EA on the slide is exactly equal to nine graduations of the scale; and since this is divided into ten equal parts, each graduation on the slide is $\frac{9}{10}$ of each graduation on the scale; hence *the difference between one graduation on the scale and one graduation on the slide is $\frac{1}{10}$ of a scale graduation.* But DA, the fraction of a graduation which we wish to measure, is the difference between CD and BA, *i.e.* between four graduations on the scale and four graduations on the slide, and is therefore $\frac{4}{10}$ of a scale graduation.

The best method of practising with the planimeter is to draw squares, triangles, and circles on a piece of paper, to measure their areas with the planimeter, and then test the results obtained, by calculating the areas measured.

199. Another method often employed for the practical determination of an area is to trace the figure accurately on a sheet of well-made tinfoil or stiff paper. The figure is then cut out from the sheet, and accurately weighed.

A rectangle of convenient size is next cut from the figure and weighed. The area of the rectangle is the product of its length and breadth ; and the area of the figure can then be determined by proportion.

For example, if the weight of the figure whose area is required be 13·6 gms., and if a rectangle cut out from it measures 8 cms. by 6 cms., and weighs 8·8 gms. ; then

$$\text{wt. of rectangle} : \text{wt. of figure} = \text{area of rectangle} : \text{area of figure}$$

$$\therefore \quad 8{\cdot}8 : 13{\cdot}6 = 8 \times 6 : x$$

$$\text{whence } x = 13{\cdot}6 \times (8 \times 6) \div 8{\cdot}8 \text{ sq. cms.}$$

CHAPTER XXI.

ON THE GRAPHICAL USE OF SQUARED PAPER.

200. On Co-ordinates.—The position of a point on a figure is most easily described by referring it to two given straight lines which are at right angles to one another. This is called the method of "**Cartesian co-ordinates,**" and is illustrated in Fig. 98. OX and OY are the lines of reference, and are called the "**co-ordinate axes;**" OX is called the **axis of** x, OY is called the **axis of** y. The point O is called the "**origin**" of the co-ordinates. The position of any point on the

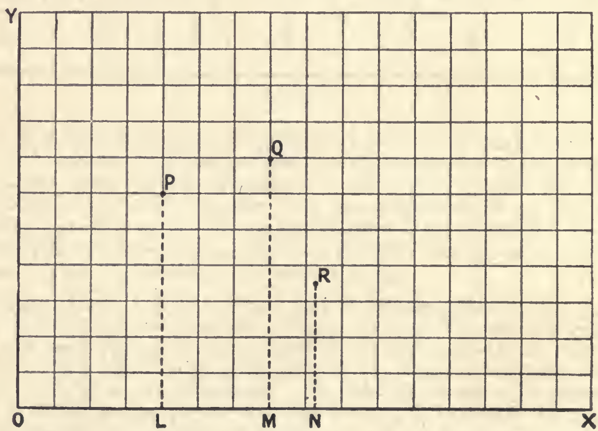

FIG. 98.

diagram is then determined by drawing the perpendicular from that point to OX, and by measuring the length of this perpendicular, and also the distance of the foot of the perpendicular from O. The perpendicular drawn from a point to OX is called the "**ordinate**" of the point; the distance of the foot of this perpendicular from O is called the "**abscissa**" of the point. Thus the ordinate and abscissa of P are PL and OL; the ordinate and abscissa of R are RN and ON.

The abscissa and ordinate are called the "**two co-ordinates**" of the point; the abscissa is called the "*x* co-ordinate," because it is measured along the axis of *x*; the ordinate is called the "*y* co-ordinate," because it is measured parallel to the axis of *y*; also these two lengths are respectively represented by the letters *x* and *y*.

If in this figure each side of each square measures $\frac{1}{2}$ cm., then OL = 2 cms., PL = 3 cms.; thus at the point P, $x = 2$ cms., $y = 3$ cms. We describe P as "the point (2, 3)." Similarly, we describe Q as "the point ($3\frac{1}{2}$, $3\frac{1}{2}$)," and R as "the point (4·1, 1·7)."

Note that if a point lies on the axis of *x*, its ordinate is zero; if it lies on the axis of *y*, its abscissa is zero; at the origin the ordinate and abscissa are both zero.

201. On the Graphical Representation of a Series of Observations.—By using abscissæ to denote quantities of one kind, and ordinates to denote quantities of another, we can represent **graphically** (*i.e.* by means of a diagram) the result of a series of observations.

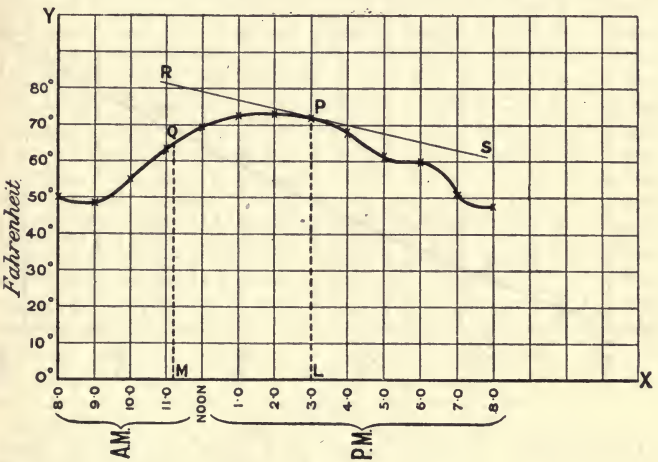

FIG. 99.

EXAMPLE (1).—*Required to draw a curve to represent the variations in temperature on a given day, the observations being as follows:* at 8 A.M., 50° F.; at 9 A.M., 48° F.; at 10 A.M., 54° F.; at 11 A.M., 63° F.; at noon, 70° F.; at 1 P.M., 72° F.; at 2 P.M., 73° F.; at 3 P.M., 72° F.; at 4 P.M., 68° F.; at 5 P.M., 61° F.; at 6 P.M., 60° F.; at 7 P.M., 51° F.; at 8 P.M., 47° F. (See Fig. 99.)

The vertical lines are marked 8 A.M., 9 A.M., etc.; the horizontal lines marked 10° F., 20° F., etc.

We then represent *each observation by a point* on the diagram; the *abscissa* of the point represents the *time which has elapsed since eight o'clock*, on the scale of one hour to a half-centimetre; the *ordinate* of the point represents *the Fahrenheit reading* on the scale of 10° to a half-centimetre. Thus to represent the observation 72° F. at 3 P.M., we measure off the abscissa **OL** to represent the interval of time between 8 A.M. and 3 P.M., and the ordinate **LP** to represent 72° F.

Having thus represented each of these observations by a point on the diagram, we draw by hand the curve which seems to pass most naturally through these points. This curve represents the variation of the temperature; it shows at a glance *at what parts of the day the temperature was rising and falling, and whether the change was rapid or gradual;* it will generally give a fairly correct value for the temperature at any time between two of the actual observations. Thus if we want to know the temperature at 11.12 A.M., we measure off the abscissa **OM** to represent 3 hours 2 minutes; the ordinate **MQ**, drawn at **M**, represents the required temperature, which is therefore 65° F.

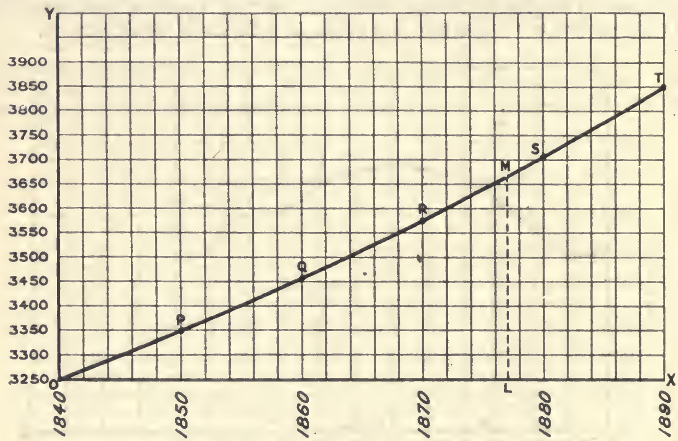

FIG. 100.

EXAMPLE (2).—*The population of a town in the year* 1840 *was* 3250; *in* 1850 *it was* 3350; *in* 1860, 3460; *in* 1870, 3580; *in* 1880, 3710; *in* 1890, 3850. *Draw a curve showing the variation of the population, and find the probable population in* 1877. (See Fig. 100.)

The abscissæ must represent the lapse of time since 1840; a convenient scale of representation will be two years to each division of the paper. The ordinates must represent the excess of the population above 3250; a convenient scale of representation will be 50 for each division of the paper. This determines the numbering of the horizontal and vertical lines as in the diagram.

The six records of population, which are given above, are then represented by the six points **O, P, Q, R, S, T** in the diagram; and the curve drawn through them represents the rate of increase of the population.

The point **L** represents the year 1877, and to determine the population in that year we use the ordinate **LM**. Working roughly, the position of **M** on the diagram is seen to represent a population of about 3670. Working accurately, by measuring we find the length of **LM** to be 4·2 cms. ; but in the ordinates, each centimetre (which is equal to two divisions) represents an increase of population of 100 ; hence **LM** represents an increase of 420. Thus the required population is $3250 + 420 = 3670$.

It is essential to notice that this diagram differs from most of the others, as *neither axis represents a zero reading;* the axis **OY** does not represent the year 0, and the axis **OX** does not represent a population 0.

EXAMPLES.—CXX.

1. The estimated value of a piece of land has undergone the following variations : in the year 1800 it was £2500 ; in 1810, £1500; in 1820, £1500; in 1830, £2400; in 1840, £3900; in 1850, £5500 ; in 1860, £6200 ; in 1870, £6200 ; in 1880, £5400. Draw a curve to represent the variation of its value, and determine the probable value at the following dates: 1835, 1847, 1858, 1875.

2. The following observations are made on the speed of a vessel in calm weather : when the engines register 100 horse-power the speed is 5·5 knots ; at 175 H.P., the speed is 11·7 knots ; at 250 H.P., 15·4 knots ; at 300 H.P., 16·9 knots ; at 400 H.P., 18·3 knots. Draw a curve representing the variation of the speed with the power employed, and find the probable speed for the following horse-powers : 150, 290, 350.

3. The following observations are made with regard to the time taken by a bullet weighing 1 oz. to fall varying distances: to fall 16 feet takes 1 second ; to fall 36 feet takes $1\frac{1}{2}$ seconds ; to fall 49 feet takes. $1\frac{3}{4}$ seconds ; to fall 64 feet takes 2 seconds ; to fall 100 feet takes $2\frac{1}{2}$ seconds. Determine by means of a diagram how long it will take the same bullet to drop the following distances : 25 feet, 56 feet, 81 feet.

4. If a ball is allowed to roll from rest down a certain slope for 2 seconds, it travels 8 feet; if for 2·4 seconds, it travels 11·5 feet; if for 3 seconds, it travels 18 feet ; if for 3·6 seconds, it travels 25·9 feet : how far will the body travel if allowed to roll for (i.) 2·8 seconds, (ii.) 3·2 seconds ?

202. On the Variation of an Algebraic Function.—Suppose that we are given some "**function of** x" (*i.e.* some algebraical expression containing x) ; and suppose that we wish to show what change is effected in the value of the function by a change in the value of x. This is most easily done by a graphical representation.

EXAMPLE.—*Represent graphically the variations in the value of the function* $\frac{x}{5}+\frac{10}{x}$, *while x increases from* 1 *to* 20.

(*a*) We find the value of $\frac{x}{5}+\frac{10}{x}$ for a series of values of x between 1 and 20.

Working correct to two decimal places—

if $x = 1, \frac{x}{5}+\frac{10}{x} = 10{\cdot}2$; if $x = 3, \frac{x}{5}+\frac{10}{x} = 3{\cdot}93$

if $x = 5, \frac{x}{5}+\frac{10}{x} = 3{\cdot}0$; if $x = 7, \frac{x}{5}+\frac{10}{x} = 2{\cdot}83$

if $x = 9, \frac{x}{5}+\frac{10}{x} = 2{\cdot}9$; if $x = 12, \frac{x}{5}+\frac{10}{x} = 3{\cdot}23$

if $x = 16, \frac{x}{5}+\frac{10}{x} = 3{\cdot}83$; if $x = 20, \frac{x}{5}+\frac{10}{x} = 4{\cdot}5$

(*b*) Representing the value of x by the abscissæ, and the values of $\frac{x}{5}+\frac{10}{x}$ by the corresponding ordinates, we plot out on the diagram the points which represent the above values of $\frac{x}{5}+\frac{10}{x}$; these are the points P, Q, R, etc. (See Fig. 101.)

The curve through these points represents the variation of the function. From the shape of the curve we see that at first the function rapidly decreases in value; that it decreases less and less rapidly till *it reaches a minimum value of about* 2·8, when x is about 7; and then increases very slowly. It should be noticed that in this example there is no "scale of representation," or, to be more correct, the scale of representation is 1 to one division on the paper.

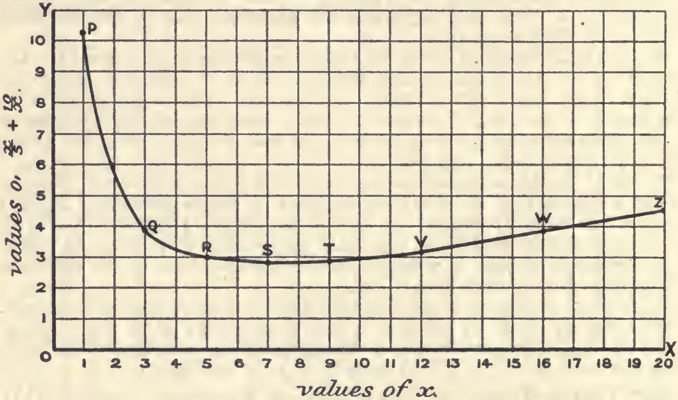

FIG. 101.

This is not necessary; nor is it necessary that the same scale of representation should be used in ordinates as in abscissæ. The scales of representation should be chosen with a view to producing a figure of convenient size; the figures should as a rule be at least twice the linear dimensions of Fig. 101.

203. It is obvious that the curve in the last paragraph is so constructed that the ordinates x and y of any point on the curve satisfy the equation $y=\frac{x}{5}+\frac{10}{x}$. The curve is then said to be "**represented by**" this equation.

Another way of expressing this is to say that the curve is the "**locus**" of those points whose co-ordinates satisfy the equation $y=\frac{x}{5}+\frac{10}{x}$; and sometimes the curve is called the "**graph**" of the function $\frac{x}{5}+\frac{10}{x}$.

EXAMPLE.—*Trace the locus* $y=\frac{10}{x^2}+x\log x$ *from* $x=1$ *to* $x=10$, *and find the minimum value of y between these limits.*

(*a*) Working correctly to two decimal places, when $x=1, y=10$; when

$x = 3$, $y = 2{\cdot}54$; when $x = 5$, $y = 3{\cdot}89$; when $x = 8$, $y = 7{\cdot}38$; when $x = 10$, $y = 10{\cdot}1$.

(*b*) If we plot out on the diagram (see Fig. 102) the points **P, Q, R, S, T,** corresponding to the above values of y, it is not easy to describe the true shape of the *first part of the curve* from these points; hence we work out the value of y when $x = 2$, which gives $3{\cdot}10$; this determines the point **Z** on the locus. The curve is now easily drawn, and shows a minimum value of about $2{\cdot}25$ for y when x is about $2{\cdot}7$.

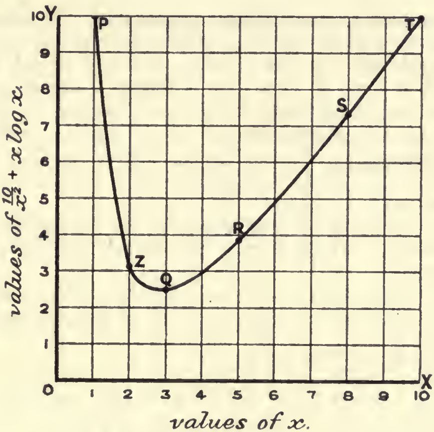

FIG. 102.

EXAMPLES.—CXXI.

Trace the following curves, from $x = 1$ to $x = 10$, and find the maximum or minimum values between those limits:—

1. $y = \frac{x}{2} + \frac{6}{x}$. **2.** $y = \frac{x^2}{12} + \frac{6}{x}$. **3.** $y = x(12 - x)$. **4.** $y = x(32 - 3x)$.

5. $y = {\cdot}2x^{\frac{3}{2}} + 5x^{-1}$. (Use logarithms in the calculations for $x^{\frac{3}{2}}$, working correct to two decimal places.)

Trace the following curves between the values $x = 0$ and $x = 3$, and determine their shape:—

6. $y = x^2$. Result: a curve through the origin and through the point $(1, 1)$; convex to the axis of x; the axis of x is a tangent; ordinates increasing.

7. $y = x$. Result: a straight line through the origin and through the point $(1\ 1)$.

8. $y = x^{\frac{1}{2}}$. Result: a curve through the origin and through the point $(1, 1)$; concave to the axis x; the axis of y is a tangent; ordinates increasing.

9. $y = x^0$. Result: a straight line through the point $(1, 1)$; parallel to the axis of x.

10. $y = x^{-1}$. Result: a curve through the point $(1, 1)$; convex to both axes; ordinates at first very large; ordinates decrease rapidly at first, less and less rapidly after.

11. $y = \frac{1}{2}x^{\frac{1}{2}}$. Result: a curve of the same type as in Question 8, passing through the origin and the point $(1, \frac{1}{2})$; every ordinate is half the corresponding ordinate in Question 8.

12. $y = \frac{1}{3}x^2$. Result: a curve of the same type as in Question 6, passing through the origin and the point $(1, \frac{1}{3})$; every ordinate is one-third of the corresponding ordinate in Question 6.

204. To calculate Logarithms to base 10.—The following method of calculating logarithms by elementary mathematics has been suggested independently by Mr. Edser, A.R.C.S.,* and by Professor Perry, D.Sc., F.R.S.† It forms an excellent example of the use of squared paper, and the student should work through it very carefully.

By extracting the square root of 10, the square root of the result, and so on, we obtain $10^{\frac{1}{2}} = 3{\cdot}1623$, $10^{\frac{1}{4}} = 1{\cdot}7783$, $10^{\frac{1}{8}} = 1{\cdot}3336$, $10^{\frac{1}{16}} = 1{\cdot}1548$, $10^{\frac{1}{32}} = 1{\cdot}0746$.

It follows from the definition of logarithms (§ 114) that—

$\frac{1}{2} = \log_{10} 3{\cdot}1623$, $\frac{1}{4} = \log_{10} 1{\cdot}7783$, $\frac{1}{8} = \log_{10} 1{\cdot}3336$, $\frac{1}{16} = \log_{10} 1{\cdot}1548$, $\frac{1}{32} = \log_{10} 1{\cdot}0746$.

Again, by multiplying these powers of 10 in various combinations, we can obtain any power of 10 whose index is an integral multiple of $\frac{1}{32}$.

For example, let us find $10^{\frac{20}{32}}$, $10^{\frac{21}{32}}$, $10^{\frac{22}{32}}$.

$$10^{\frac{20}{32}} = 10^{\frac{5}{8}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{8}} = 3{\cdot}1623 \times 1{\cdot}3336 = 4{\cdot}2170$$
$$10^{\frac{21}{32}} = 10^{\frac{20}{32}} \times 10^{\frac{1}{32}} = 4{\cdot}2170 \times 1{\cdot}0746 = 4{\cdot}5316$$
$$10^{\frac{22}{32}} = 10^{\frac{20}{32}} \times 10^{\frac{2}{32}} = 10^{\frac{20}{32}} \times 10^{\frac{1}{16}} = 4{\cdot}2170 \times 1{\cdot}1548 = 4{\cdot}8697$$

Thus, writing these indices as logarithms—

$$\left.\begin{aligned} \log 4{\cdot}2170 &= \tfrac{20}{32} = {\cdot}625 \\ \log 4{\cdot}5316 &= \tfrac{21}{32} = {\cdot}65625 \\ \log 4{\cdot}8697 &= \tfrac{22}{32} = {\cdot}6875 \end{aligned}\right\} \mathbf{A}$$

Proceeding in this manner, we can get a complete table of all the numbers whose logarithms are $\frac{1}{32}$, $\frac{2}{32}$, $\frac{3}{32}$, etc. . . . $\frac{30}{32}$, $\frac{31}{32}$. Also $0 = \log 1$, and $1 = \log 10$.

If we now take a sheet of squared paper and plot these results, representing numbers by abscissæ and logarithms by ordinates, we could draw a curve through the points so obtained which would enable us to estimate very roughly the logarithm of any quantity between 1 and 10, or the antilogarithm of any quantity between 0 and 1. This is obviously the curve $y = \log x$, between the values $x = 1$ and $x = 10$.

We can, however, obtain a considerable degree of accuracy if we plot three consecutive results only on a large diagram. We will use the three consecutive results worked above, and marked **A**.

Take a large sheet of good squared paper. Choose for origin a point near the left-hand lower corner.

* See "Measurement and Weighing" (Chapman & Hall, 1899), p. 59.

† "Practical Mathematics" (Eyre & Spottiswoode), p. 39.

Graduations along **OX** are to represent numbers between 4·217 and 4·8697. Mark **O** as 4·20; the next division should represent 4·21, the next 4·22, and so on. Every fifth division along **OX** should be marked. As we wish to represent the number 4·8697, this will require that **OX** should contain at least 67 divisions.

Graduations along **OY** are to represent logarithms between ·625 and ·6875. Mark **O** as ·62; the next division should represent ·622, the next ·624, and so on. Every fifth division along **OY** should be marked; and **OY** must contain at least 34 divisions.

Now plot points **P, Q, R**, representing as accurately as possible the three results in **A**. It will be found that **P, Q, R** are nearly in a straight line. A flat ruler can easily be bent into a curve to contain these three points.

The curve traced in this manner will indicate the logarithm of any quantity between 4·2 and 4·9, or the antilogarithm of any logarithm between ·62 and ·69. Both logarithms and anti-logarithms can be estimated to four significant figures with very fair accuracy.

205. On Negative Co-ordinates.—It is often required that abscissæ or ordinates should represent negative quantities. A con-

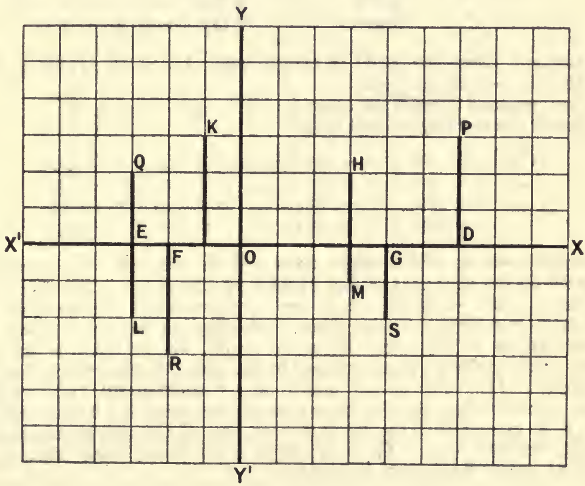

FIG. 103.

sistent method of representation to include both positive and negative quantities is furnished by the following

RULE. — **Abscissæ representing positive quantities are measured to the right of the origin; abscissæ representing negative quantities are measured to the left of the origin. Ordinates representing positive quantities are drawn above**

the axis of x; ordinates representing negative quantities are drawn below the axis of x.

Thus in Fig. 103, where O is the origin of co-ordinates, and XOX′, YOY′ are the axes—

The co-ordinates of P are (+OD, +PD), or measuring them in divisions of the paper (+6, +3).

The co-ordinates of Q are (−OE, +QE), *i.e.* (−3, +2), for OE represents a negative quantity, since it lies to the left of O.

The co-ordinates of R are (−OF, −FR), *i.e.* (−2, −3), for OF represents a negative quantity, since it lies to the left of O, and FR represents a negative quantity, since it lies below the axis of x.

The co-ordinates of S are (+OG, −GS), *i.e.* (+4, −2), for GS represents a negative quantity, since it lies below the axis of x.

The four regions XOY, YOX′, X′OY′, Y′OX, into which the two co-ordinate axes divide the diagram, are called the first, second, third, and fourth "**quadrants**" respectively. In consequence of the above rule, the following statements are self-evident :—

For a point in the first	quadrant,	XOY,	x is +, y is +
" " second	"	YOX′,	x is −, y is +
" " third	"	X′OY′,	x is −, y is −
" " fourth	"	Y′OX,	x is +, y is −

EXAMPLE.—*Draw the graph of the function* $x^3 - 2x^2 - 24x + 25$ *between the values* $x = -6$ *and* $x = +8$.

We are required to trace the curve $y = x^3 - 2x^2 - 24x + 25$.

By actual calculation, we find that—

when $x = -6, y = -119$;	when $x = -4, y = +25$
" $x = -2, y = +57$;	" $x = 0, y = +25$
" $x = +2, y = -23$;	" $x = +4, y = -39$
" $x = +6, y = +25$;	" $x = +8, y = +217$

The values of x, which range from −6 to +8, can be conveniently represented on the scale of 1 to one division of the paper. The values of y, which range from −119 to +217, can be conveniently represented on the scale of 20 to one division of the paper. Plotting on these scales, we obtain the graph shown in Fig. 104. From the graph, we see that the value of y increases till the point F, then decreases to the point G, after which it increases continuously. The value of y is said to be a "**maximum**" at F (although there are greater values of y in the region of the point Z), because the value of y at F is greater than the value of y at any of the neighbouring points. Similarly, the value of y at G is said to be a minimum, being (algebraically) less than at any of the surrounding points.

206. On the Solution of Equations.—By means of the graph of the preceding paragraph, we can solve the equation $x^3 - 2x^2 - 24x + 25 = 0$; for y represents the function $x^3 - 2x^2 - 24x + 25$, and y is zero at the points C, D, and E; but at C the value of x is approximately −4·5, at D it is +1, and at E it is +5·5 approximately. Hence the above equation is satisfied by either of these values of x.

The values of x which satisfy an equation are called the "**roots**" of the equation. We have learnt in Chapter IX. how to solve "simple" equations algebraically—simple equations being those which contain only the first power of the unknown letter. Simple equations

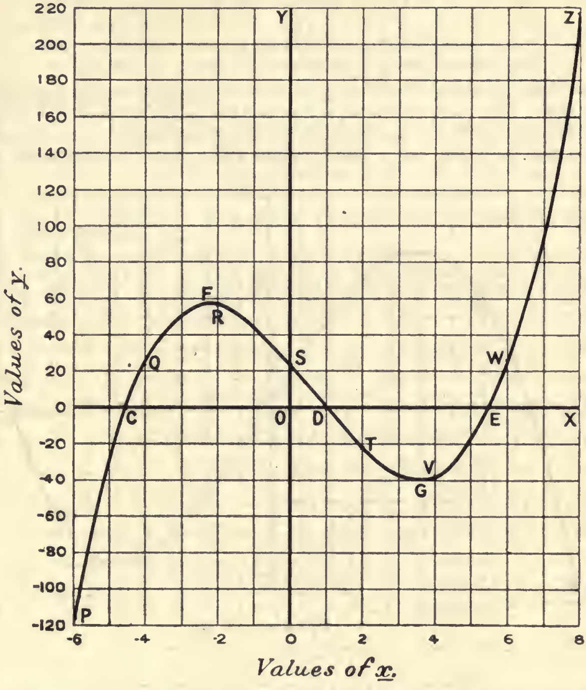

Fig. 104.

never have more than one root, but other equations *may* have more than one root.

From these considerations we derive the following

RULE.—**To solve an equation, simplify it (if necessary), and carry all terms to the left-hand side of the equation; trace the graph of the function on the left-hand side; the roots of the equation are the values of *x* at the points where the graph cuts the axis of *x*.**

EXAMPLE.—*Find any roots of the equation* $x^{2 \cdot 096} = 90 - \frac{11}{x}$, *which lie between* -10 *and* $+10$.

Clearing of fractions: $x^{3 \cdot 096} = 90x - 11$

whence $x^{3 \cdot 096} - 90x + 11 = 0$

We next trace the curve $y = x^{3.096} - 90x + 11$; the values of $x^{3.096}$ must be estimated by logarithms.

when $x = -10, y = -348$; when $x = -7, y = +224$
,, $x = -4, y = +298$; ,, $x = -2, y = +183$
,, $x = 0, y = 11$; ,, $x = +2, y = -161$
,, $x = +4, y = -277$; ,, $x = +7, y = -202$
,, $x = +10, y = -370$

From these values we draw the graph of the function, which is represented in Fig. 105. The roots of the equation are the values of x at the points **A**, **B**, and **C**, viz. $-8{\cdot}6$, $+{\cdot}12$, $+8{\cdot}5$. To solve this equation accurately, the graph should be drawn on a much larger scale than is represented in this

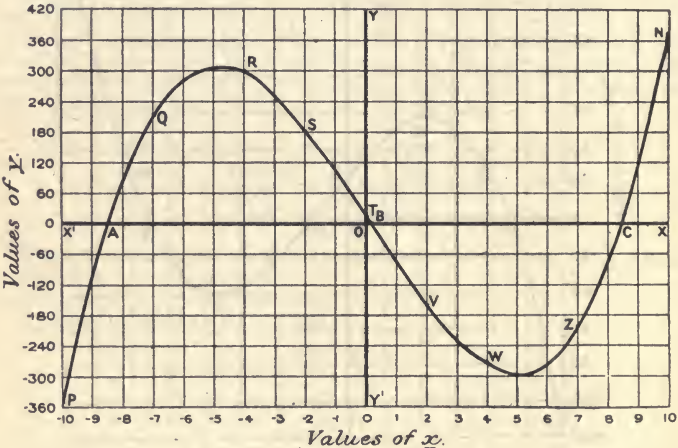

FIG. 105.

figure, representing x on a scale of about 1 to three divisions of the paper, and y on a scale of about 20 to one division. Also more values of y should be calculated in the neighbourhood of **A**, **B**, and **C**.

EXAMPLES.—CXXII.

1. Write down the co-ordinates of the points **H, K, L, M**, in Fig. 103, measuring them in divisions of the paper.

Find the roots of the following equations, which lie between the given values of x :—

2. $2x^2 - 19x + 35 = 0$, between 0 and 10.

3. $6x^2 + \frac{170}{x} = 71$, between 0 and 10.

4. $2x^2 + 3x - 20 = 0$, between -5 and 5.

5. $x^2 + 20 = 10x$, between -3 and 9.

6. $x + 5 = \frac{48}{x}$, between -10 and 10.

7. $x^{2 \cdot 1} + 2x^{1 \cdot 1} = 48$, between 0 and 10.

8. $3^x = 10x$, between 0 and 10.

9. $27(x + 10)^{-1 \cdot 1} = 10 \log (x + 10)$, between -8 and 2.

Trace the following curves between the values $x = -3$ and $x = 3$, and determine their shape :—

10. $y = e^{\cdot 5x}$. Result : a curve entirely above XOX′, and convex to XOX′ ; cutting OY at a distance 1 from O ; ordinates increase continually.

11. $y = -e^{\cdot 5x}$. Result : the reflection of the preceding curve in a mirror placed in the position XOX′.

12. $y = e^{-\cdot 5x}$. Result : the reflection of the curve of Question 10, in a double-faced mirror placed in the position OY.

13. $y = -e^{-\cdot 5x}$. Result : the reflection of the curve of Question 12, in a mirror placed in the position XOX′.

14. $y = e^x$. Result : a curve of the same type as in Question 10 ; intersecting OY at the same point.

207. The method of the preceding paragraph at best does not determine the roots of the equation correctly to more than two significant figures. But by taking a minute portion of the curve in the

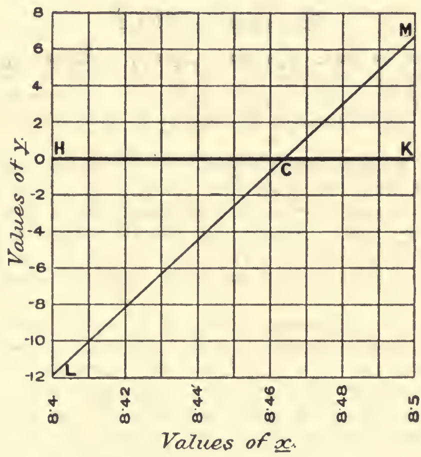

FIG. 106.

neighbourhood of a point where it cuts the axis of x, and enlarging it, we can determine the root correctly to three significant figures.

For example, if we wish to determine the root represented by the point C in Fig. 105 correctly to three significant figures, we find by calculation that when $x = 8{\cdot}5$, the actual value of y is $6{\cdot}7$; since the curve trends upwards * at C, it follows that 8·5 is rather too large a

* For the negative values of y occur before the point C, and the positive values after.

value for the root. We then calculate the value of y when $x = 8{\cdot}4$, and obtain $-11{\cdot}7$. We now represent the two points which correspond to these values of y, *on any convenient scales of representation*, as in Fig. 106. In this figure HK represents a small portion of the axis of x in the neighbourhood of the point C in Fig. 105, while L and M are the two neighbouring points on the curve which we have now determined. (The scale of representation of x is 100 times that of the preceding figure, while the scale of representation of y is thirty times that of the preceding figure.) *Since* LM *represents a very minute portion of the curve, we may regard it as practically a straight line;* this gives a very close approximation for the true position of C. The figure shows that the value of x at C, correct to three significant figures, is $8{\cdot}46$.

We could obtain a very high degree of accuracy by evaluating the values of y correctly to four significant figures, when x has the values $8{\cdot}4$, $8{\cdot}5$, and $8{\cdot}6$. If these three results are plotted on a large diagram, the slightly curved line which passes through the points obtained can be drawn by using a slightly bent flat ruler.

EXAMPLES.—CXXIII.

Solve, correct to three significant figures, equations 5, 6, 7, 8, 9 of Example CXXII.

CHAPTER XXII.

ON CORRECTION OF ERRORS, AND RATES OF INCREASE.

208. On Straight-line Graphs.—If the student will trace for himself the graphs represented by the following equations, he will find that in each case the graph is a straight line :—

$$y = 3 + 2x \quad \text{(i.)}$$
$$y = 2{\cdot}5 + {\cdot}15x \quad \text{(ii.)}$$
$$y = -2 - x \quad \text{(iii.)}$$
$$y = 6{\cdot}3 - {\cdot}35x \quad \text{(iv.)}$$

It is, in fact, a general law that the **graph represented by an equation of the type $y = a + bx$** (where a and b represent any *constant* quantities, positive or negative) **will always be a straight line.**

The reason of this law can be roughly explained as follows :—

It is obvious that the slope of a graph at any point depends on the rate of increase of y with regard to x. For instance, in Fig. 101, § 202, the curve at P is sloping steeply downwards to the right, which shows that the value of y is rapidly decreasing ; while at T the curve slopes gently upwards to the right, showing that the value of y is slowly increasing.

Now, consider the first of the above equations, $y = 3 + 2x$; whenever we increase the value of x by 1, we increase the value of $2x$ by 2, and therefore we increase the value of $3 + 2x$, *i.e.* of y, by 2. Thus it follows that the increase of y is always twice the increase of x; or that in this graph *the rate of increase of y with regard to x is always the same*. Hence, *the slope of the graph must be the same at all points*, which is only possible when the graph is a straight line.

In the same way, if we take the fourth of the above equations, $y = 6{\cdot}3 - {\cdot}35x$, it is clear that whenever we increase the value of x by 1, we *diminish* the value of y by ·35. Hence the *rate of decrease of y is constant;* and therefore the graph must *trend downwards*, and its slope must be the same at all points ; the graph will therefore be a straight line sloping downwards to the right.

Notice also that, in the equation $y = a + bx$, when $x = 0, y = a$; and when x is increased by 1, y is increased by b.

We thus obtain the following

Theorem.—The graph represented by the equation $y = a + bx$

(where a and b are any constants) is always a straight line. It slopes upwards if b is positive; downwards if b is negative. The quantity b represents the change in the value of y when the value of x is increased by unity. The quantity a represents the value of y when x is zero.

209. On finding the Equation to a given Straight-line Graph.—We are now in a position to find the equation to any graph which is a straight line by the application of the theorem of the last paragraph.

EXAMPLE (1).—*Find the equation to the straight-line graph which passes through the points* (2, 3), (6, 5).

Mark on a diagram the points **P**, **Q** at (2, 3) and (6, 5) respectively. (See Fig. 107.) Draw the line through **P** and **Q**, cutting the axis **OY** at **R**. Now, by the above theorem the equation $y = a + bx$ represents the graph **PQ**, provided a represents the value of y when $x = 0$, and b represents the change in the value of y when x is increased by unity. But when $x = 0$ the value of y is represented by **OR**, which is 2; thus $a = 2$. To find accurately the change in the value of y when x is increased by unity, it is best to compare two points at a fair distance from one another, such as **R** and **Q**. In passing from **R** to **Q**, the increase of x is measured by **RS**, and is therefore 6; the

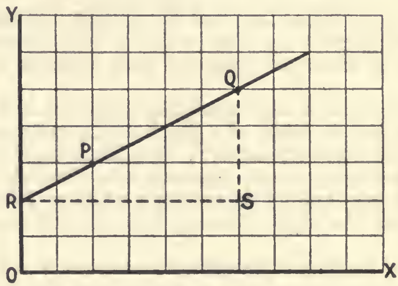

FIG. 107.

increase of y is measured by **QS**, and is therefore 3. Thus when x increases by 6, y increases by 3; therefore when x increases by 1, y increases by ·5; thus $b = \cdot 5$. Hence the required equation is $y = 2 + \cdot 5\,x$.

EXAMPLE (2).—*Find the equation to the graph of Fig.* 108.

In order that $y = a + bx$ should represent this graph, $a =$ the value of y when x is zero $=$ **OP** $= 4{\cdot}6$; and to find b we may conveniently compare the points **L** and **M**. In passing from **L** to **M** the increase of $x =$ **LN** $= 5$, and the *decrease* of $y =$ **NM** $= 1{\cdot}6$. Thus when x increases by unity, the *decrease* of $y = 1{\cdot}6 \div 5 = \cdot 32$; thus remembering that b is negative, *because the line slopes down*, we have $b = -\cdot 32$. Thus the required equation is $y = 4{\cdot}6 - \cdot 32x$.

210. To determine the Equation to the Straight-line Graph through Two given Points.—If the co-ordinates of two points on a graph are given, we can determine its equation *without using a* *diagram.* There is some advantage in this method, because the accuracy of the method of the last paragraph depends entirely on the accuracy of drawing and of measuring.

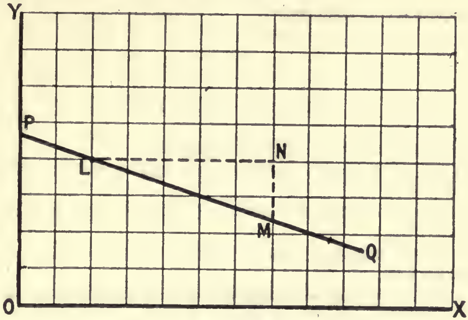

FIG. 108.

The following examples explain the method :—

EXAMPLE (1).—*Find the equation to the graph through the points* (2, 3) *and* (6, 5).

Let $y = a + bx$ be the required equation. We have to find the values of a and b. Since the point (2, 3) is on the graph, its equation is satisfied when $x = 2, y = 3$.

$$\text{Hence} \quad 3 = a + 2b \quad \ldots \ldots \quad \text{(i.)}$$

Similarly, since the point (6, 5) is on the graph—

$$5 = a + 6b \quad \ldots \ldots \quad \text{(ii.)}$$

The values of a and b are obtained from equations (i.) and (ii.) by the method of § 84.

Subtracting equation (i.) from equation (ii.)—

$$5 - 3 = a + 6b - a - 2b$$
$$\textit{i.e.} \quad 2 = 4b$$
$$\text{whence} \quad b = \cdot 5$$

Substituting this value of b in equation (i.), we have—

$$3 = a + 1$$
$$\text{which gives} \quad a = 2$$

Hence the required equation is—

$$y = 2 + \cdot 5x$$

EXAMPLE (2).—*Find the equation to the straight-line graph through the points* (2, 3·5) *and* (5, ·8).

Let $y = a + bx$ be the required equation. As in the preceding example, we obtain—

$$3\cdot 5 = a + 2b \quad \ldots \ldots \quad \text{(i.)}$$
$$\cdot 8 = a + 5b \quad \ldots \ldots \quad \text{(ii.)}$$
$$\text{subtracting} \quad 3\cdot 5 - \cdot 8 = a + 2b - a - 5b$$
$$\textit{i.e.} \quad 2\cdot 7 = -3b$$
$$\text{whence} \quad b = -\cdot 9$$

Substituting this value of b in equation (i.), we obtain—

$$3{\cdot}5 = a - 1{\cdot}8$$

whence $a = 5{\cdot}3$

Thus the required equation is—

$$y = 5{\cdot}3 - {\cdot}9x$$

EXAMPLES.—CXXIV.

Find the equation to the straight-line graph—

1. Which cuts off a length 3 from **OY**, and in which y increases by 3 when x increases by 2.

2. Which cuts off a length 2 from **OY′**, and in which y increases by 2 when x increases by 5.

3. Which cuts off a length 3 from **OY**, and in which y decreases by 1 when x increases by 10.

4. Which cuts off a length 1 from **OY′**, and in which y decreases (algebraically) by 1 when x increases by 20.

5. Which passes through **O**, and in which y increases by 3 when x increases by 5.

6. Which passes through **O**, and in which y increases by 2 when x increases by 7.

7. Which passes through the points (2, 2) and (5, 4).

8. Which passes through the points (1, 3) and (5, 4).

9. Which passes through the points (1, 5) and (4, 2).

10. Which passes through the points (2, ·4) and (5, ·3).

11. Which passes through the points (0, ·3) and (4, ·2).

12. Which passes through the points (0, ·6) and (6, ·5).

13. Draw the straight-line graph which passes through the points (2, 3) and (4·8, 3·7); and from the graph determine the equation.

14. Draw the straight-line graph which passes through the points (15, 23) and (4, 19); and from the graph determine the equation.

15. Draw the straight-line graph which passes through the points (8, 20) and (15, 11·6); and from the graph determine the equation.

16. Draw the straight-line graph which passes through the points (10, 12) and (17, 12·35); and from the graph determine the equation.

211. On the Determination of a Law and the Correction of Errors.—When we know or suspect that two quantities (represented by x and y) are connected by an equation of the type $y = a + bx$, we often determine the probable values of a and b by a series of experiments and a diagram. The method is best illustrated by examples.

EXAMPLE (1).—*A weight is suspended at the extremity of an elastic string.* Representing the stretched length of the string in inches by y, and the weight in ounces by x, and assuming that they are connected by a law of the type $y = a + bx$, determine the probable values of a and b from the following record of observations: when $x = 5$, $y = 6{\cdot}75$; when $x = 9$, $y = 7{\cdot}7$; when $x = 13$, $y = 8{\cdot}4$; when $x = 16$, $y = 9{\cdot}4$; when $x = 18{\cdot}5$, $y = 10{\cdot}3$.

In Fig. 109 we mark the points **H, K, L, M, N,** which correspond to the above values for x and y. These points do not lie accurately in a straight line; hence either the law connecting them is not accurately of the type $y = a + bx$, or else there have been slight errors in the observations. Adopting the latter view, we proceed to determine the straight line which *appears to pass most evenly between these points.* This is done by stretching a piece of fine thread across the diagram till its line appears to be in correct position. In this case the line **PQ** appears to be the correct line. We then take **PQ** to represent the graph of the required law; and its equation will represent the law. To find the equation to **PQ** we use the method of § 209, thus $a = \text{OP} = 5{\cdot}3$; also the "gradient" is found conveniently by comparing the points **R** and **S**, and is therefore a gradient of **ST** in **RT**, *i.e.* 5 in 19, *i.e.* ·263 in 1; whence $b = {\cdot}263$. Thus the required law is $y = 5{\cdot}3 + {\cdot}263x$. Having obtained the law, we can find the (probably) correct value of y for any value of x; thus putting $x = 16$, $y = 5{\cdot}2 + {\cdot}263 \times 16 = 9{\cdot}5$. Or we may read the value of y corresponding to any value of x from the graph itself.

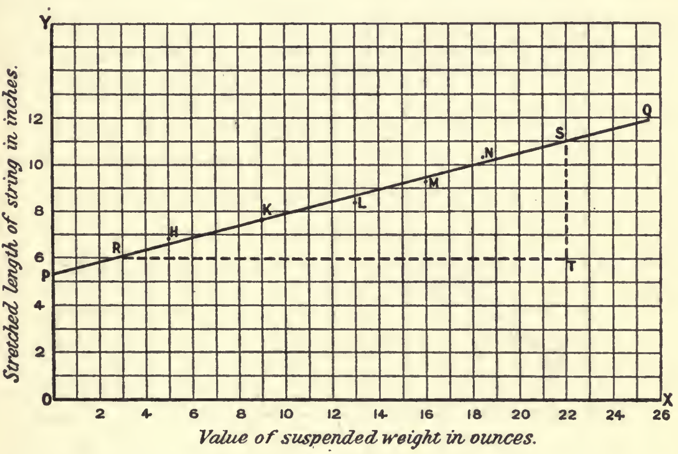

FIG. 109.

EXAMPLE (2).—*Find the probable pressure of a gas in a closed cylinder when the temperature is* 200° *Centigrade, given the following observations; also determine the probable law which connects the pressure and the temperature. At a temperature of* 60° *C. the pressure was* 13·8 *lbs. per sq. inch; at* 90° *C.,* 14·85 *lbs. per sq. inch; at* 120° *C.,* 16·2 *lbs. per sq. inch; at* 130° *C.,* 16·75 *lbs. per sq. inch; at* 150° *C.,* 17·5 *per sq. inch.*

In Fig. 110 the points **P, Q, R, S, T** correspond to the above-recorded observations, and the line **LZ** appears to pass most evenly between them. Thus the probable pressure at 200° C., which is represented by the point **Z,** is 19·5 lbs. per sq. inch.

To determine the equation to this graph, we cannot use the method of § 209, because the *diagram does not show* the value of y when $x = 0$. But we can use the method of § 210 by taking any two points on the line, for instance **V** and **Z.** At **V,** $x = 70$, $y = 14{\cdot}14$; at **Z,** $x = 200$, $y = 19{\cdot}5$.
Thus to find the values of a and b, we have the equations—

$$14{\cdot}14 = a + 70b \quad \text{(i.)}$$
$$19{\cdot}5 = a + 200b \quad \text{(ii.)}$$

whenc $a = 11{\cdot}6$, $b = {\cdot}0412$. Thus the required equation is—

$$y = 11{\cdot}6 + {\cdot}0412x$$

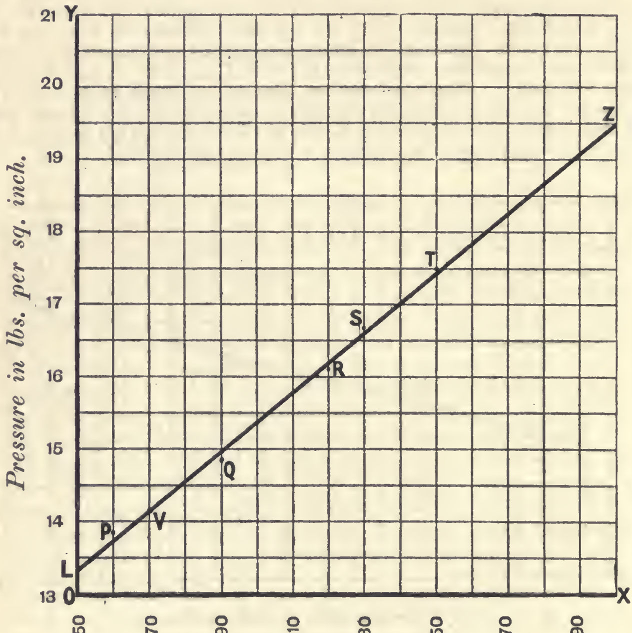

FIG. 110.

EXAMPLES.—CXXV.

1. Two quantities x and y are known to be connected by a relation of the type $y = a + bx$. The following are the observed values of y for the given values of x:—

x =	20	40	48	80	100
y =	97	152	168	251	302

Draw the straight-line graph which corresponds most closely to these observations; find its equation, and determine the probable values of y which correspond to the following values of x: (i.) 30, (ii.) 50, (iii.) 100.

2. Two quantities P and Q are known to be connected by a relation of the type $Q = a + bP$. The following table records observed pairs of corresponding values of P and Q :—

P =	2	4·5	8	10·4	12	15
Q =	5·57	6·38	7·41	8·1	8·58	9·53

Find the probable values of a and b, and determine the probable values of Q when P is 8 and when P is 12; also determine the probable values of P when Q is 7·7 and when Q is 6·95.

3. Two quantities x and y are connected by a law of the type $y = a + bx$. Deduce from the following recorded observations the probable value of y when x is 1200, and the probable values of x when y is 50, and when y is 60; also find the probable values of a and b :—

x =	1100	1120	1150	1180	1220	1250	1300
y =	64·3	62·1	59·2	57·1	53	50·3	46·2

4. Experiments are made on the amount of stretching which takes place in a steel bar when subjected to varying tensions—

Tension in lbs.-wt. =	140	400	500	720	840	1000
Amount of stretching in inches =	·3	·8	·98	1·45	1·7	1·97

Draw the straight-line graph which appears to correspond most closely with these observations. Find its equation; find the probable stretching when the tension is 300 lbs.-wt., and also when it is 650 lbs.-wt.; and find the probable tension when the stretching is ·45 inch.

5. The following observations are made with regard to the horse power at which a locomotive engine is working when travelling at 40 miles an hour with varying loads behind it :—

Load in tons =	50	65	70	85	90	100
Rate of working in H.P. =	81	96	106·5	120·5	127	142

Draw the straight-line graph which corresponds most nearly to these observations; find its equation; and give the probable values of the horse power when the load is 70 tons, and when it is 80 tons.

6. A steel rod is supported at its two extremities. The following observations are made with regard to the depression of the middle of the rod below the extremities when different weights are suspended from the middle points :—

Weight in grammes	= 100	120	140	150	180	200
Depression in centimetres	= 1·13	1·27	1·39	1·49	1·76	1·88

Assuming that there is a law of the type $y = a + bx$ connecting these quantities, where x represents the weight and y the depression, find the probable values of a and b, and the probable value of the depression when the weight is 50 grammes.

212. On Determining the Law which connects Two Quantities from a List of Observations.—There is an endless variety in the types of equations which connect two co-varying quantities; it is therefore essentially a matter of guess-work and luck when we try to discover the equation connecting two quantities x and y from a series of recorded observations.

The method of procedure is to try to plot, from the recorded observations, a series of points in a straight line. Suppose we first plot (as usual) values of x as abscissæ, and values of y as ordinates; then if the points so obtained lie in a straight line, the required equation is of the form $y = a + bx$, and is determined as in § 209. If this method fails, we may plot the values of $\frac{1}{x}$ as abscissæ, and values of y as ordinates; if these points lie in a straight line, the required equation is of the form $y = a + b \cdot \frac{1}{x}$, and is determined as in § 209.

Similarly, *if we obtain a straight line by plotting any functions of x and y as abscissæ and ordinates*, the method of § 209 at once determines the required equation. For example—

Abscissæ.	Ordinates.	Type of equation when the plotted points lie in a straight line.
$\frac{1}{x}$	$\frac{1}{y}$	$\frac{1}{y} = a + b \cdot \frac{1}{x}$
x	y^2	$y^2 = a + bx$
x^3	y	$y = a + bx^3$
$\log x$	y	$y = a + b \cdot \log x$
x	$\frac{y}{x}$	$\frac{y}{x} = a + bx$
$\frac{x}{y}$	$\log y$	$\log y = a + b \cdot \frac{x}{y}$

EXAMPLE.—*Find the law connecting the quantities x and y, given that when x has the values* 1, 4, 6, 9, 16, *y has the values* 8, 6, 5·1, 4, 2 *respectively.*

Plotting the values of x and y, we do not obtain a straight line. But if we plot the values of $\sqrt{x}$ as abscissæ, and the values of y as ordinates, we obtain Fig. 111, where the points P, Q, R, S, T lie in a straight line.

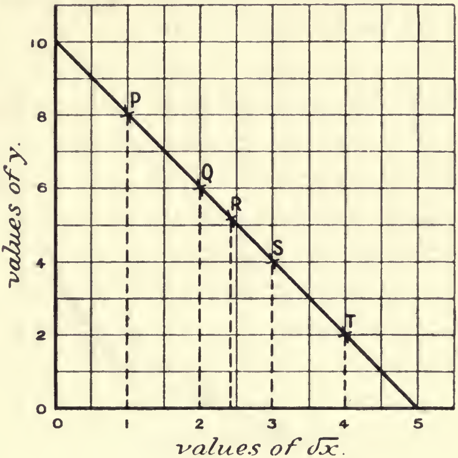

FIG. 111.

The required equation is therefore of the type $y = a + b\sqrt{x}$, and the constants a and b are determined as in § 209; thus drawing the straight line, we see that when $\sqrt{x}$ is 0, y is 10. Also comparing the points P and Q, when $\sqrt{x}$ increases by 1, y decreases by 2. Thus we obtain the equation—

$$y = 10 - 2\sqrt{x}$$

Much careful and intelligent practice is necessary for success in this process. At the present stage the student may be content if he understands how to derive the equation, when told how to obtain the straight-line graph.

EXAMPLES.—CXXVI.

Determine the laws which connect x and y from each of the following sets of observations :—

1. $\left.\begin{array}{l} x = 1, 2, 3, 6 \\ y = 8, 5, 4, 3 \end{array}\right\}$

(Plot $\frac{1}{x}$ and y.)

2. $\left.\begin{array}{l} x = 0, 1, 2, 2{\cdot}83 \\ y = {\cdot}5, {\cdot}625, 1, 1{\cdot}5 \end{array}\right\}$

(Plot x^2 and y.)

3 $\left.\begin{array}{l} x = 0, 1, 2{\cdot}6, 3{\cdot}4 \\ y = 1{\cdot}73, 2{\cdot}83, 4, 4{\cdot}67 \end{array}\right\}$

(Plot x and y^2.)

4. $\left.\begin{array}{l} x = 0, 5, 10, 15 \\ y = {\cdot}2, {\cdot}1, {\cdot}067, {\cdot}05 \end{array}\right\}$

(Plot x and $\frac{1}{y}$.)

5. $\left.\begin{array}{l} x = 0, 2, 4, 6 \\ y = 12, 6, 4, 3 \end{array}\right\}$

(Plot xy and y.)

6. $\left.\begin{array}{l} x = 1, 2, 3, 4 \\ y = 4{\cdot}5, 8, 10{\cdot}5, 12 \end{array}\right\}$

(Plot x and $\frac{y}{x}$.)

213. On Rates of Increase in Curved Graphs.—We have previously stated that the slope of a graph at any point represents the rate of increase (or decrease) of y with regard to x at that point.

Now, we measure the slope of a curve at any point by the *slope of the tangent* to the curve at that point; for example, in Fig. 99, § 202, the slope of the curve at the point P is the same as the slope of the tangent RS which touches the curve at P.

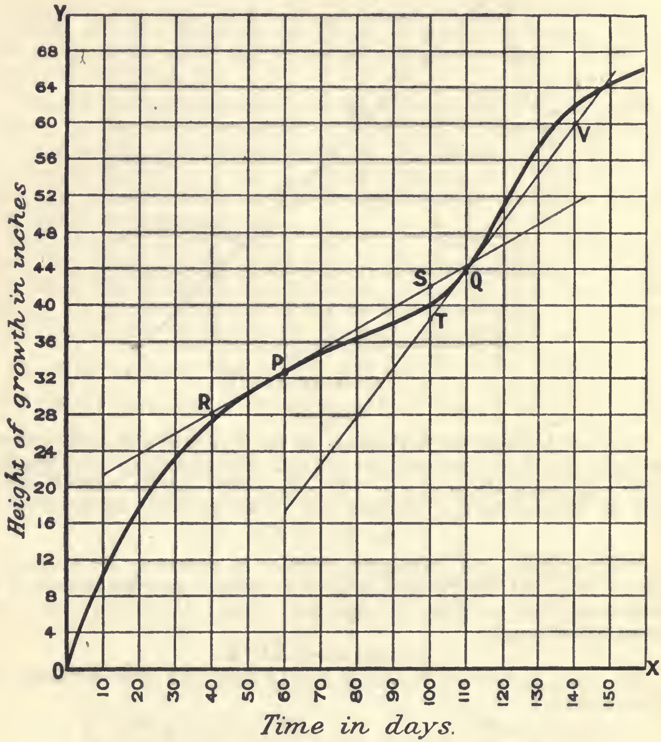

Fig. 112.

Example (1).—*The curve in Fig.* 112 *represents the growth of a plant during* 160 *days; the ordinates represent the height on the scale of* 4 *inches to one division of the paper; and the abscissæ represent the time of growth on the scale of* 10 *days to one division: find the rate of growth after* 60 *days, and also after* 110 *days.*

The point P on the curve corresponds to the sixtieth day. The rate of growth is represented by the slope of the curve at P, which is the same as the slope of the tangent at **P**; that is to say, *the rate of growth at* **P** *is the same as that of a plant whose growth is represented by the straight-line graph* **RS.** We choose any two convenient points **R** and **S** on this straight line (not too close together), and from them we estimate the rate of growth. **R** represents a height of 28 inches on the fortieth day; **S** represents a height of 42 inches on the 100th day; thus **RS** shows an increase of 14 inches in height in a period

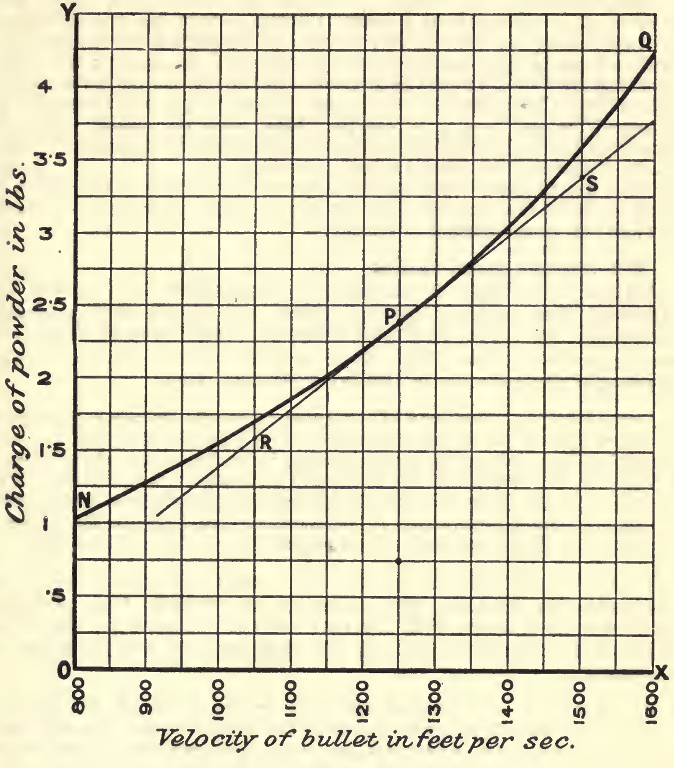

FIG. 113.

of 60 days. Thus *the rate of growth* along **RS** is $\frac{7}{30}$ inch per day, which may be taken as the actual *rate of growth* on the sixtieth day.

In the same way, the point **Q** corresponds to the 110th day. Choosing two convenient points **T** and **V** on the tangent at **Q**, we see that **T** represents a height of 38 inches on the 100th day, and **V** a height of 60 inches on the 140th day; this gives an increase of 22 inches in 40 days, or ·55 inch per day.

EXAMPLE (2).—*If the curve in Fig.* 113 *represents the relation between the muzzle velocity of the bullet from a certain gun, and the charge of powder required to produce that velocity, how do you interpret the slope of the curve at* **P**?

In this curve the abscissæ show muzzle velocities, and the ordinates show the charges of powder required to produce them. Also the point P corresponds to a muzzle velocity of 1250 feet per second. Thus the slope of the curve at P shows the relation between *a small increase of charge and the small increase of muzzle velocity produced when the muzzle velocity is about* 1250 *feet per second.*

Now, the slope of the curve at P is the same as that of the tangent RS. The point R would represent a muzzle velocity of 1050 feet per second produced by a charge of 1·6 lbs. of powder; and the point S would represent a muzzle velocity of 1500 feet per second produced by a charge of 3·4 lbs. Thus comparing these two, we have an increase of 450 feet per second in the velocity produced by an increase of 1·8 lbs. in the charge; this is *at the rate of* 1 foot per second for each ·004 lb. of charge. Thus when the muzzle velocity is about 1250 feet per second, the extra charge required to increase it is about ·004 lb. to each foot per second increase of velocity.

Note that at higher velocities the slope of the curve increases; thus at higher velocities the increase of charge is at a higher rate than ·004 lb. for each foot per second increase of velocity.

214. Some Special Cases.

RULE.—**If a body is moving in a straight line, and if we represent the motion by a graph in which the abscissa represents the time, and the ordinate the distance from the starting-point; then the slope of the curve at any point represents the velocity of the body at that time.**

EXPLANATION.—The slope of the curve represents the rate of increase of the distance from the starting-point with regard to the time; but if at any instant the distance from the starting-point is increasing at the rate of 5 feet in each second, it follows that the velocity of the body at that instant must be 5 feet per second. Note also that if the curve slopes downwards at any point, the distance from the starting-point is diminishing (provided that the ordinates are positive). Hence the body is then returning toward the starting-point.

RULE.—**If a body is moving in a straight line, and if we represent its motion by a graph in which the abscissa represents the time, and the ordinate the velocity; then the slope of the tangent represents the rate of acceleration or retardation.**

EXPLANATION.—The slope of the curve represents the rate of increase of the velocity, which is the rate at which the body's motion is accelerated. Where the curve slopes downwards, the slope represents the rate of decrease of the velocity, which is the rate at which the body's motion is retarded.

EXAMPLE (1).—*Fig.* 114 *represents the motion of a body which returns to the starting-point after* 11 *seconds. The abscissa represents the time, and the ordinate the distance from the start. What is the position of the body after* $3\frac{1}{2}$ *seconds and after* 9 *seconds? and what is its velocity at these times?*

The point P represents the position of the body after $3\frac{1}{2}$ seconds, and shows that the distance from the starting-point is then 6 feet. The slope of the curve at P represents the velocity at this instant, and is measured by the slope of the tangent at P. Since R represents a distance of 3 feet after 1 second, and P a distance of 6 feet after $3\frac{1}{2}$ seconds, the line RP represents a change of position of 3 feet in $2\frac{1}{2}$ seconds; this represents a velocity of $3 \div 2\frac{1}{2}$, *i.e.* of

1·2 feet per second. Since the curve slopes upwards at **P**, the body is then moving away from the starting-point.

The position of the point **Q** on the diagram shows that at the end of 9 seconds the distance from the starting-point is 6 feet 2 inches. The velocity is given by the slope of the tangent at **Q**; since the point **T** represents a distance of 8 feet from the starting-point at the end of 8 seconds, and the point

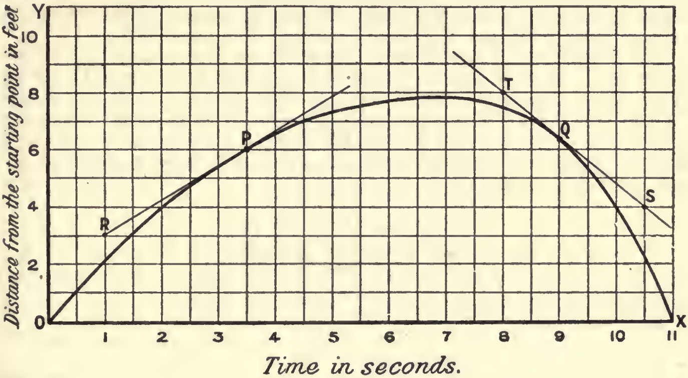

FIG. 114.

S represents a distance of 4 feet at the end of 10·5 seconds, the line **TS** represents a change of position of 4 feet in $2\frac{1}{2}$ seconds; this gives a velocity of $4 \div 2\frac{1}{2}$, *i.e.* 1·6 feet per second. Since the curve slopes downwards at **Q**, the body is then moving back toward the starting-point.

EXAMPLE (2).—*Fig.* 115 *represents the motion of a train which starts from one station, attains a velocity of* 30 *miles an hour, and comes to rest at the next*

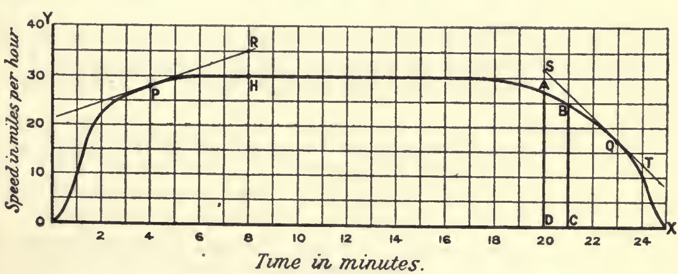

FIG. 115.

station in 25 *minutes. The abscissæ represent times, and the ordinates velocities. Find from the curve the velocity and the rate of change of velocity at the end of the fourth, eighth, and twenty-third minutes.*

The position of the point **P** shows that the velocity at the end of the fourth minute is 28 miles per hour. The rate of increase of velocity is shown by the slope of the tangent **PR** ; since **P** represents a velocity of 28 miles per hour at the end of the fourth minute, and **R** a velocity of 35 miles per hour at the end of the eighth minute, the line **PR** represents an increase of velocity of 7 miles an hour in 4 minutes ; thus the rate of increase of velocity is 1·75 miles per hour per minute.

The position of the point **H** shows that the velocity at the end of the eighth minute is 30 miles per hour ; and since the graph in the neighbourhood of **H** is straight and parallel to **OX**, it is obvious that the velocity is neither increasing nor decreasing at the end of the eighth minute.

The position of the point **Q** shows that at the end of the twenty-third minute, the velocity is 17 miles per hour. The rate of *decrease* of velocity at **Q** is represented by the slope of the tangent **ST**. **S** represents a velocity of $31\frac{1}{2}$ miles per hour at the end of 20 minutes, and **T** a velocity of $12\frac{1}{2}$ miles per hour at the end of 24 minutes ; thus the line **ST** represents a decrease of velocity of 19 miles an hour in 4 minutes ; thus the rate of decrease of velocity is 4·75 miles per hour per minute.

EXAMPLES.—CXXVII.

1. Draw a graph through the following points: (0, 0), (2, 4·5), (4, 6·3), (6, 7·7), (8, 8·9), (10, 10). Find the rate of increase of y with regard to x at the points where $x = 3$, and where $x = 8$.

2. Draw a graph through the following points : (0, 1), (2, 1·4), (4, 2·6), (7, 5·9), (10, 11), (12, 15·4). Find the rate of increase of y with regard to x at the points where $x = 5$, and where $x = 11$.

3. An iron bar is freely hinged at one end, and is supported in a horizontal position by a vertical string tied to some other point of the bar. The following table shows the variation of the tension in the string, when the string is fastened at different distances from the hinge :—

Distance from the hinge in inches	2	4	6	8	9	12
Tension of the string in lbs.-wt.	14·4	7·2	4·8	3·6	3·2	2·4

Draw a graph representing the variation of the tension ; and find the rate of decrease of the tension as the string is moved farther from the hinge, (i.) at a distance of 5 inches from the hinge, (ii.) at a distance of 7 inches from the hinge.

4. Water is flowing uniformly through a pipe into a vessel in the shape of an inverted cone ; the following table shows the depth of water in the vessel at certain intervals after the flow commences :—

After the lapse of	5 secs.	10 secs.	15 secs.	20 secs.	25 secs.	30 secs.
The depth of water is	6·84 ins.	8·62 ins.	9·87 ins.	10·86 ins.	11·7 ins.	12·43 ins.

Draw the corresponding graph ; and find the rates of increase of level, (i.) after 8 seconds, (ii.) after 27 seconds, (iii.) when the depth of water is 10 inches.

5. If the following table gives the height of the barometer at different heights above the sea-level, under certain atmospheric conditions :—

Height above sea-level	0	ft. 5000	ft. 10,000	ft. 20,000	ft. 30,000	ft. 40,000	ft. 50,000	ft. 60,000
Height of barometer	ins. 30	ins. 24·9	ins. 20·6	ins. 14·2	ins. 9·8	ins. 6·7	ins. 4·6	ins. 3·2

Draw a graph to represent the variation of the barometric height; and determine the rate of decrease at a height of 20,000 feet, and at a height of 40,000 feet.

6. The following table represents the distances from the starting-point of a moving body after the lapse of the given intervals of time :—

When the number of seconds elapsed is	1	2	3	4	5	6	7	8	9
The distance in feet from the starting-point is	18	32	42	48	50	48	42	32	18

Determine, by means of a graph, the velocity of the body at the end of the third and at the end of the eighth second.

7. The following table gives the distances of a body from its starting-point after the lapse of the given intervals of time :—

When the body has been moving	1 sec.	2 secs.	3 secs.	4 secs.	6 secs.	8 secs.
The distance travelled is	3·3 ft.	26·6 ft.	90 ft.	213 ft.	720 ft.	1707 ft.

Find the velocity at the end of the third second, and at the end of the sixth second.

8. The following table gives the velocities of a body at different times :—

After	1 sec.	2 secs.	4 secs.	6 secs.	9 secs.	12 secs.
The velocity in feet per second is	55	50	40	30	15	0

Find the rate of change of velocity.

9. The velocity of a body varies in accordance with the following table :—

After the lapse of	1 min.	2 mins.	3 mins.	4 mins.	5 mins.	6 mins.
The velocity in miles per hour is	10	17·3	20	17·3	10	0

Find the rate of change of velocity at the end of the second minute, and at the end of the fifth minute.

215. On Rates of Increase by Calculation.—If we know the equation to a graph—that is to say, the law connecting the quantities represented in the graph—we may determine the rate of increase at any point by direct calculation. To do this we compare the values of x and y at the given point with the values of x and y at a point *very close* to the given point.

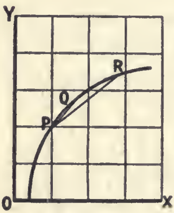

FIG. 116.

The explanation of this can be seen from Fig. 116. In this figure the slope of the line PR differs considerably from the slope of the curve at P, *i.e.* from the slope of the tangent at P. But the slope of the line PQ, which joins two points which are fairly close together on the curve, differs but little from the slope of the curve at P; and it is obvious that if Q is moved nearer and nearer to P, the slope of the line PQ differs less and less from the slope of the curve at P.

EXAMPLE (1).—*Find by calculation the rate of increase of y with regard to x at the point* R *in the graph of Fig.* 102, § 203.

The equation to this curve is $y = \frac{10}{x^2} + x \log x$. At R, $x = 5$—

$$\therefore y = \frac{10}{25} + 5 \log 5 = 3{\cdot}89485$$

We now take a value of x slightly larger than 5, say 5·01; then—

$$y = \frac{10}{(5{\cdot}01)^2} + 5{\cdot}01 \log (5{\cdot}01) = {\cdot}398449 + 5{\cdot}01 \times {\cdot}6998377 = 3{\cdot}904636$$

Thus $x = 5{\cdot}01$, $y = 3{\cdot}904636$, *are the co-ordinates of a point very close to* R. Thus we see that if x increases by ·01, y increases by ·009786. *At this rate of increase*, if x increased by 1, y would increase by ·9786, which is approximately the required rate of increase of y with regard to x at the point R.

This is not the exact rate of increase at R; for, although these points are exceedingly close together, the line joining them has not *exactly* the same slope as the tangent.

The *exact* determination of the rate of increase requires the use of the Differential Calculus. But the above method will usually give the result *correct to two significant figures*, provided that the two values of x *differ by* 1 *in the third significant figure*. The less the difference between the two values of x, the more accurate will be the result.

This method is more accurate than the method of § 213, because (*a*) it is impossible in drawing to represent the *exact* shape of the curve; (*b*) it is impossible to ensure that the tangent drawn is the tangent at the given point,—it is more likely to be the tangent at a point very close to the given point; (*c*) the measurements by which the slope of the tangent is calculated are at best only correct to three significant figures, and are liable to error, owing to slight inaccuracies in the ruling of the squared paper.

On the other hand, we cannot use the method of this paragraph *unless we know the equation to the graph.*

EXAMPLE (2).—*If the details of the motion of a body are correctly given by the equation,* $s = 16t^2 - \cdot 05t^3$, *where s represents the distance travelled in feet, and t the time from the start in seconds, find the velocity of the body after* 8 *seconds.*

Note that in this example t and s represent the same quantities as the x and y co-ordinates in Example (1) of § 207; hence the rate of increase of s with regard to t measures the velocity of the body.

When $t = 8$, $s = 16t^2 - \cdot 05t^3 = 16 \times 64 - \cdot 05 \times 512 = 998\cdot 4$; when $t = 8\cdot 01, s = 16 \times (8\cdot 01)^2 - \cdot 05 \times (8\cdot 01)^3 = 1026\cdot 5616 - 25\cdot 6961 = 1000\cdot 8655$. That is to say, after 8 seconds the body has travelled 998·4 feet; after 8·01 seconds it has travelled 1000·8655 feet; comparing these two results, we see that the rate of travelling at this time is 2·4655 feet in ·01 second, or 250 feet per second, correct to two significant figures. (The actual rate, determined by the Calculus, is 246·4 feet per second.)

EXAMPLES.—CXXVIII.

1. The equation to a graph is $y = \frac{x}{5} + \frac{10}{x}$; find the rate of change of y with regard to x, (a) when $x = 16$, (b) when $x = 12$, (c) when $x = 3$.

2. The equation to a graph is $y = 5x - \cdot 6x^2$; find the rate of change of y with regard to x, (a) when $x = 1$, (b) when $x = 4$, (c) when $x = 7$.

3. The equation to a graph is $y = \cdot 3x + 6x^{-1}$; find the rate of change of y with regard to x, (a) when $x = 2$, (b) when $x = 4$, (c) when $x = 10$.

4. If the distance (in feet) from the starting-point be represented by s, and the time (in seconds) by t, and if the law governing the motion be $s = 10t - \cdot 05t^3$, determine the velocity, (a) at the end of 5 seconds, (b) at the end of 10 seconds.

5. If the velocity (in inches per second) be represented by v, and the time (in seconds) by t, and if the law connecting these quantities be $v = 5t - 6t^{\frac{1}{2}}$, find the rate of change of velocity, (a) at the end of the third second, (b) at the end of the sixteenth second.

216. On the Symbol $\frac{dy}{dx}$ and its Value when $y = ax^n$.—In the preceding paragraphs we have been determining the "rate of increase of y with regard to x." In the Calculus we use the symbol $\frac{dy}{dx}$ to denote this rate of increase. Note particularly that $\frac{dy}{dx}$ *does not mean* $\frac{d \times y}{d \times x}$, but is the symbol for a rate of increase. This rate of increase is also called the "**differential coefficient**" of y with regard to x, and may also be expressed by the symbol $\left(\frac{d}{dx}\right)y$.

Now, the value of $\frac{dy}{dx}$ obviously depends on the law of the graph, *i.e.* on the equation which connects y with x; and in the Calculus we learn to determine $\frac{dy}{dx}$ for any type of equation.

The following useful formula should be committed to memory :—

$$\text{If } y = ax^n, \frac{dy}{dx} = nax^{n-1}$$

Note that this formula is only true when a and n are constants.

ILLUSTRATIONS.—

$$\text{If } y = 3x^2, \frac{dy}{dx} = 2 \times 3x = 6x$$

$$\text{If } y = 5x^3, \frac{dy}{dx} = 3 \times 5x^2 = 15x^2$$

$$\text{If } y = x^{5.7}, \frac{dy}{dx} = 5{\cdot}7x^{4.7}$$

$$\text{If } y = 2x^{-3.2}, \frac{dy}{dx} = (-3{\cdot}2) \times 2x^{-3.2-1} = -6{\cdot}4x^{-4.2}$$

EXPLANATION.—We can prove this law when $n = 2$ and when $n = 3$. The other cases are beyond the scope of this book.

(i.) If $y = ax^2$. Suppose that x increases very slightly to $(x + h)$, so that h is a very small quantity. (Cf. the method of § 215.) Then y increases to $a(x + h)^2$.

Thus the increase of $y = a(x + h)^2 - ax^2 = ax^2 + 2axh + ah^2 - ax^2 = 2axh + ah^2$.

Thus when x increases by h, y increases by $2axh + ah^2$; therefore at the same rate—

If x increased by 1, y would increase by $(2axh + ah^2) \div h = 2ax + ah$

Thus the rate of increase of y with regard to x is $2ax + ah$ approximately.

But just as in § 215 this approaches nearer and nearer to the correct value as h becomes smaller. Hence we get the correct value by reckoning h as zero, when $2ax + 2ah = 2ax$.

$$\therefore \frac{dy}{dx} = 2ax$$

(ii.) If $y = ax^3$. By the same method we see that if x increases by h, y increases by $a(x + h)^3 - ax^3 = ax^3 + 3ahx^2 + 3ah^2x + ah^3 - ax^3 = 3ahx^2 + 3ah^2x + ah^3$.

Thus the rate of increase of y with regard to x is—

$$(3ahx^2 + 3ah^2x + ah^3) \div h = 3ax^2 + 3ahx + ah^2$$

This gives the correct value, if we reckon $h = 0$.

$$\text{Thus } \frac{dy}{dx} = 3ax^2$$

EXAMPLE (1).—*Find $\frac{dy}{dx}$, when $y = 4x^{1.5}$. Hence determine the rate of increase of y with regard to x when $x = 2$, and when $x = 4$.*

$$\frac{dy}{dx} = 1{\cdot}5 \times 4\,x^{1.5-1} = 6x^{.5}$$

$$\therefore \text{if } x = 2, \frac{dy}{dx} = 6 \times 2^{.5} = 6 \times 1{\cdot}414 = \mathbf{8{\cdot}484}$$

$$\text{Also if } x = 4, \frac{dy}{dx} = 6 \times 4^{.5} = 6 \times 2 = \mathbf{12}$$

Learn also the formula—

$$\text{If } y = a + bx^n + cx^m$$

$$\frac{dy}{dx} = nbx^{n-1} + mcx^{m-1}$$

In this case the preceding rule has been *applied separately to the second and third terms, bx^n and cx^m.* The first term a does not affect $\frac{dy}{dx}$; for since a does not contain x, an increase in the value of x does not affect the first term; hence the first term does not affect the rate of increase of y.

EXAMPLE (2).—*Find $\frac{dy}{dx}$, when $y = a + bx$.*

$$y = a + bx^1; \therefore \frac{dy}{dx} = 1 \times bx^{1-1} = bx^0 = b$$

Note that this result is in agreement with § 208, where we have shown that if $y = a + bx$, b denotes the change in the value of y when x is increased by unity; *i.e.* b denotes the rate of increase of y with regard to x.

EXAMPLE (3).—*Find $\frac{dy}{dx}$, when $y = 3 - 5x + 4x^3$; and give the values when $x = \cdot 5$, and when $x = 2$.*

$$y = 3 - 5x^1 + 4x^3$$

$$\therefore \frac{dy}{dx} = -1 \times 5x^{1-1} + 3 \times 4x^{3-1}$$

$$= -5x^0 + 12x^2 = -5 + 12x^2$$

Thus when $x = \cdot 5$, $\frac{dy}{dx} = -5 + 12 \times \cdot 25 = -2$

Also when $x = 2$, $\frac{dy}{dx} = -5 + 12 \times 4 = 43$

In the first case the rate of increase is negative, and therefore y is *decreasing.* In the second case the rate of increase is positive and large, and therefore y is *increasing rapidly.*

EXAMPLES.—CXXIX.

Find the values of $\frac{dy}{dx}$ (*i.e.* the rate of increase of y with regard to x) in each of the following cases, and evaluate them for the given values of x:—

1. $y = 3x^2$ $[x = 3]$.
2. $y = \cdot 5x^3$ $[x = 2]$.
3. $y = \cdot 125x^4$ $[x = 2]$.
4. $y = \cdot 214x^{-1}$ $[x = \cdot 5]$.
5. $y = \cdot 24x$ $[x = 7\cdot 2]$.
6. $y = \cdot 3x^{2\cdot 5}$ $[x = 4]$.
7. $y = \cdot 3x^{-2\cdot 5}$ $[x = 9]$.
8. $y = x^6$ $[x = 2]$.
9. $y = 64x^5$ $[x = \frac{1}{2}]$.
10. $y = -4x^{-2}$ $[x = \frac{1}{2}]$.
11. $y = \cdot 3 + \cdot 5x^2$ $[x = 1, 2]$.
12. $y = 3 + 2x^{\frac{1}{2}}$ $[x = 1, 3]$.
13. $y = \cdot 5 + \cdot 6x$ $[x = 3]$.
14. $y = 21 + 3x^{1\cdot 5}$ $[x = 5]$.
15. $y = 2 + 3x^2 - 4x^3$ $[x = \cdot 5]$.
16. $y = \cdot 2x + x^{-1}$ $[x = 2]$.
17. $y = 2\cdot 5 - 1\cdot 5x + 2x^{-\frac{1}{2}}$ $[x = \cdot 25]$.
18. $y = 20 + 3x^{\frac{1}{2}} - 2x^{-1}$ $[x = 2]$.

CHAPTER XXIII.

ON THE AREAS OF GRAPHS.

217. On the Use of Graphs for Determination of Volumes.—The following rule is useful in determining the volume of an irregular body, such as the trunk of a tree, a ship, an Indian club, etc. We first imagine the body cut into slices by a series of planes which are perpendicular to the length of the body and parallel to one another. We must know (*a*) the distance of each of these planes from one end of the body; (*b*) the area of the surface in which the plane cuts the body (which is called the "**sectional area**" of the body). Thus in Fig. 117, which represents a block of wood, the length of the block is

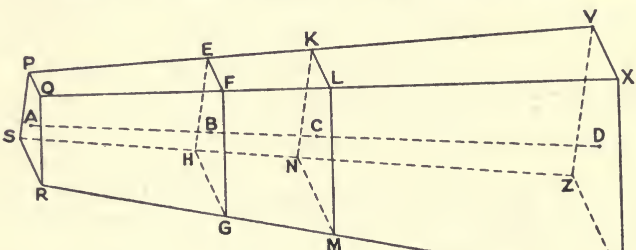

FIG. 117.

measured by AD; the "sectional area" at E is the area of the quadrilateral EFGH, formed by one of the planes at right angles to AB; the "sectional area" at K is the area of the quadrilateral KLMN, formed by another plane at right angles to AB. The distance of the plane EFGH from the end of the block is measured by AB; the distance of the plane KLMN from the end of the block is measured by AC. A large number of such planes would divide the block into slices.

RULE.—**To determine the volume of a body, draw a graph in which the ordinates represent the sectional areas at different distances from the end of the body, these distances being represented by the abscissæ. The area included between the graph and the axis of *x* will then represent the volume of the body.**

EXAMPLE.—*The sectional areas of a tree-trunk* 32 *feet high, at different distances from its base, are given by the following table:—*

Distance from the base in feet	0	4	8	14	19	22	28	32
Sectional area in square inches	444	352	280	207	158	136	110	102

Find its volume in cubic feet.

The points **P**, **Q**, **R**, etc., on Fig. 118, represent the above measurements; the distances from the base being represented by the abscissæ on the scale of

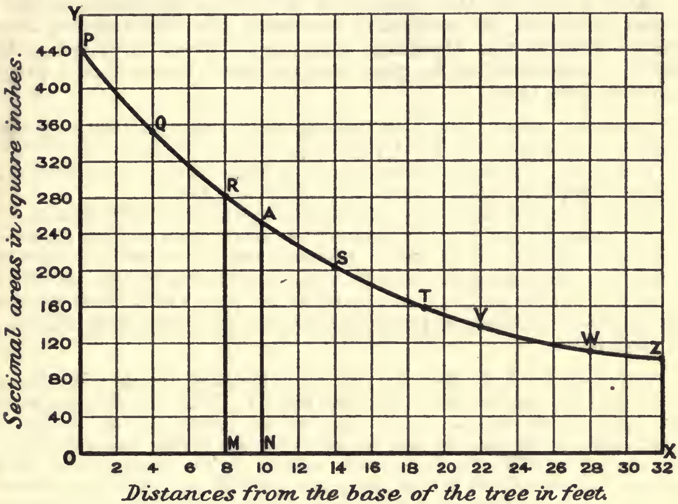

FIG. 118.

2 feet to one division of the paper; and the sectional areas being represented by the ordinates, on the scale of 40 sq. inches to one division. Then the area **OPZXO** represents the volume of the trunk.

To find the scale of representation of the volume. Taking any square of the paper, the horizontal side of the square represents 2 feet, and the vertical side represents 40 sq. inches. Thus the area of the square represents a volume 2 feet × 40 sq. inches = 24 inches × 40 sq. inches = 960 cub. inches.

We next calculate the area **OPZXO.** By Simpson's Rule, we find the area to be equivalent to 84·5 squares.* (In this case the method of counting the squares gives exactly the same result.) Since each square represents a volume of 960 cub. inches, the total volume of the trunk is 84·5 × 960 cub. inches = 47 cub. feet very approximately.

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* To apply Simpson's Rule to obtain the equivalent number of squares, we measure all lengths in terms of divisions of the paper; *e.g.* **NA** = 6·3 divisions.

EXPLANATION.—To show that the area between the curve and **OX** represents the volume of the trunk, consider the area **RMNA**; this corresponds to a slice of the trunk included between two planes at distances 8 and 10 feet respectively from the base. Now, the volume of this slice is the product of its thickness and its average sectional area; also the area of **RMNA** is the product of its base **MN** and the average height of the ordinate between **R** and **A**. But the base **MN** represents the thickness of the slice, and the average height of the ordinate represents the average sectional area of the slice. Thus the area of **RMNA** represents the volume of this slice. A similar argument obviously applies to each vertical strip of the area **OPZXO**; and therefore the whole area represents the volume of the trunk.

218. RULE.—**If the motion of a body is represented by a graph in which the ordinates represent the velocities at times represented by the abscissæ, then the distance travelled by the body is represented by the area included between the graph and the axis of** x.

EXAMPLE.—*Find the distance travelled by the train whose motion is represented by the graph of Fig.* 115, § 214.

In accordance with the above rule, the distance travelled is represented by the area **OPQXO**, included between the graph and **OX**.

By Simpson's Rule we obtain this area as equivalent to 126·2 squares.

To obtain the scale of representation of the area. If we take any square, its horizontal side represents a time of 1 minute, and its vertical side represents a speed of 5 miles per hour; thus its area represents the distance travelled in 1 minute when the speed is 5 miles per hour, *i.e.* $\frac{1}{12}$ *of a mile.* Thus the whole area, which is equivalent to 126·2 squares, represents a distance of $126·2 \times \frac{1}{12}$ $= 10·52$ miles.

EXPLANATION.—To show that the area of the graph represents the distance travelled, let us consider the strip of area **ABCD**. **DC** represents the twenty-first minute of the motion. To find the distance travelled during any time, we must multiply the time by the average velocity; in this case, the time is represented by **CD**, and the average velocity by the average value of the ordinate between **A** and **B**; but the product of **CD** and the average value of the ordinate gives the area of this strip; thus the area of this strip represents the distance travelled in the time **CD**. A similar argument applies to any other vertical strip of the area. Thus the whole area represents the whole distance travelled.

219. The Average Value of the Ordinate.

RULE.—**To find the average value of the ordinate for any portion of a graph, draw the ordinates at the extremities of that portion; calculate the area included between these ordinates, the graph, and the axis of** x**; and divide by the distance between the feet of the ordinates.**

EXAMPLE (1).—*Calculate the average value of the ordinate for points between* **A** *and* **Z** *on the graph of Fig.* 118, § 217.

We divide the area **ANXZ** by the distance **NX**. The area **ANXZ** will be found equivalent to 42 squares, and the length **NX** is 11 divisions. Thus the average value of the ordinate for points between **A** and **Z** is $42 \div 11$, *i.e.* 3·82 divisions.

The average value of this ordinate obviously corresponds to the average value of the sectional area for the upper 22 feet of the trunk; and, since each division represents 40 sq. inches, the average value of the sectional area of this part of the trunk is 3·82 × 40 sq. inches, *i.e.* 152·8 sq. inches.

EXAMPLE (2).—*Calculate (and interpret) the average value of the ordinate for the whole curve in Fig.* 115, § 214.

We divide the area **OPQXO** by the distance **OX.**

The area **OPQXO** = 126·2 squares, and the distance **OX** = 25 divisions. Dividing, we find that the average value of the ordinate is 5·048 divisions. This represents the average velocity of the train during the whole motion, and, since the scale is 5 miles per hour for each division, the average velocity is 5 × 5·048, *i.e.* 25·24 miles per hour.

Note that we could also calculate the average velocity from the result of § 218, that the whole distance travelled in the 25 minutes was 10·52 miles.

EXAMPLES.—CXXX.

1. Calculate the volume of a cone 8 inches high, given the sectional areas at stated distances from the base, as follows :—

Distance from the base in inches	0	2	4	6	8
Sectional area in square inches	24	13·5	6	1·5	0

2. Determine the volume of water contained in a well of depth 20 feet, given the following sectional areas at various depths :—

Depth in feet	0	4	8	12	16	20
Sectional area in square feet	124·7	84	51·3	26·7	10	1·3

Also determine the average sectional area.

3. A Rugby football measures 14 inches in length. If **A** represents the sectional area in square inches at a distance **D** inches from one end, and if these quantities are connected by the law, $A = \cdot 785[49 - (D - 7)]^2$; draw a graph representing the variation of the sectional area, and from it determine the volume of the football.

4. The velocities of a body at different times in its motion are given by the following table :—

Time from the start in seconds	0	1	2	3	4	6	8	10
Velocity in feet per second ...	0	1	1·6	2	2·29	2·67	2·9	3·08

Determine the distance travelled in the 10 seconds, and the average velocity.

5. From the following data, construct the graph which represents the variation of the velocity, and determine the total distance travelled in 90 seconds, and the average velocity for that time :—

Time from the start in seconds	0	10	20	30	50	70	90
Velocity in feet per second ...	100	50	33·3	25	16·7	12·5	10

6. If v represent the velocity of a body in miles per hour after t minutes from the start, and if these quantities obey the law, $y = t^2 - \cdot05t^3$; determine the distance travelled in the first 10 minutes, and the average velocity for that time; also the distance travelled in the second 10 minutes, and the average velocity for that time.

CHAPTER XXIV.

ON GEOMETRICAL PROPORTIONS, AND SIMILAR FIGURES.

220. Theorem I.—If we draw one or more lines parallel to one side of a triangle and cutting the other two sides, these two sides will be divided in the same proportion.

Thus in Fig. 119 the line DE is parallel to the side BC of the triangle ABC. The sides AB, AC are each divided in the ratio 5 : 3, as in the figure.

Thus AD : DB = AE : EC

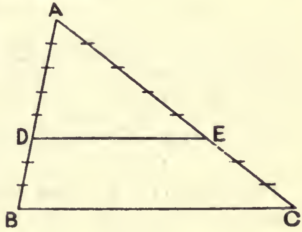


Fig. 119.

Again, in Fig. 120 the lines HK, FG, LM are each parallel to the side QR. The sides PQ, PR are each divided in the proportion 3 : 6 : 4 : 10, as shown in the figure.

Thus PH : FL = PK : GM ; HL : FQ = KM : GR

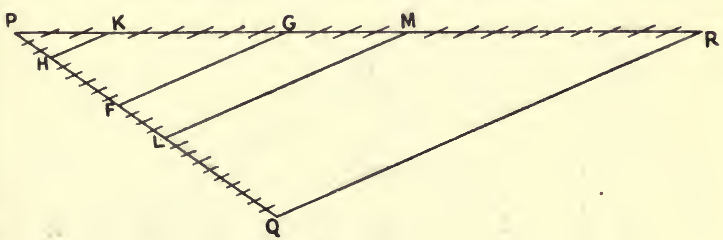


Fig. 120.

And in general the ratio of any two lengths on PQ is equal to the ratio of the two corresponding lengths on PR.

The theorem can also be extended to the case of Fig. 121, where DE and FG are each parallel to BC ; but DE cuts the sides AB and AC when produced. In this figure

EA : AF : FB = 4 : 2 : 7 = DA : AG : GC

Conversely, if in the triangle ABC (Fig. 119) we know that D and E divide the sides AB, AC in the same ratio; then we may infer that DE is parallel to BC.

Problem I.—To find the fourth proportional to three given lines, p, q, r. (See Fig. 122.)

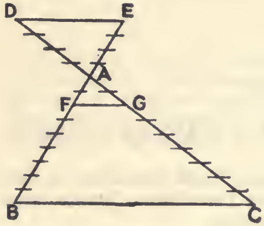


FIG. 121.

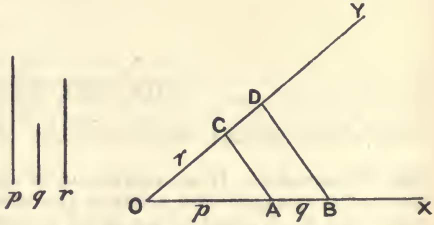


FIG. 122.

Take any two lines OX, OY at a convenient angle. On OX mark off OA $= p$, AB $= q$; on OY mark off OC $= r$. Join AC, and through B draw BD parallel to AC. The length CD will be the required fourth proportional.

For by Theorem I. since AC is parallel to BD, OA : AB = OC : CD; thus $p : q = r :$ CD; *i.e.* CD is the fourth proportional to p, q, r.

Problem II.—To find the third proportional to two given lines, p, q. (See Fig. 123.)

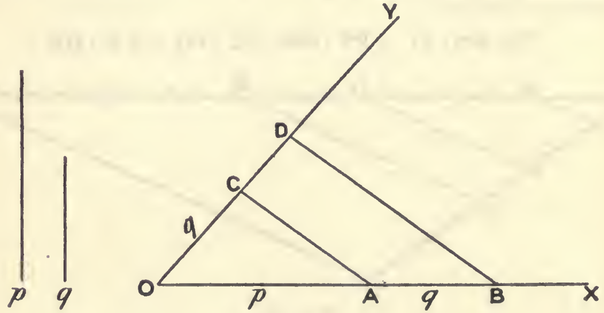


FIG. 123.

Take any two lines OX, OY at a convenient angle. On OX mark off OA $= p$, AB $= q$; on OY mark off OC $= q$. Join AC, and through B draw BD parallel to AC. The length CD will be the required third proportional.

For by Theorem I. OA : OB = OC : CD; thus $p : q = q :$ CD; *i.e.* CD is the third proportional to p and q.

Problem III.—**To divide a line into segments (*i.e.* portions) which shall be proportional to certain given lines.** (See Fig. 124.)

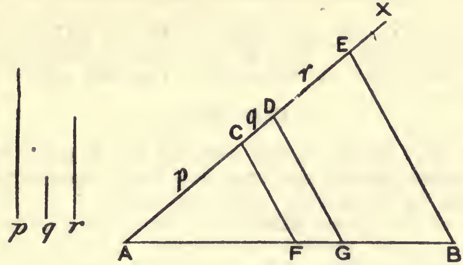


FIG. 124.

It is required to divide the line **AB** into segments proportional to the given lines *p*, *q*, *r*. Draw **AX** at any convenient angle. On **AX** mark off **AC**, **CD**, **DE** respectively equal to *p*, *q*, *r*. Join **EB**, and through **C** and **D** draw lines **CF**, **DG** parallel to **EB**. The points **F**, **G** will divide **AB** as required.

For by Theorem I. the lengths **AF**, **FG**, **GB** are proportional to **AC**, **CD**, **DE**; *i.e.* to *p*, *q*, *r*.

Problem IV.—**To divide a given line into any number of given parts.** (See Fig. 125.)

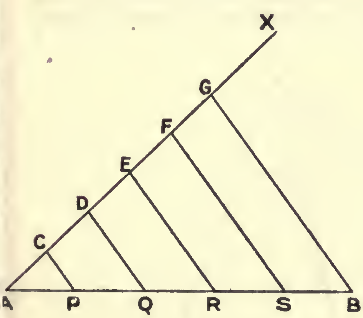


FIG. 125.

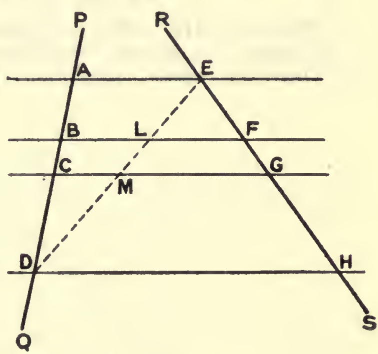


FIG. 126.

It is required to divide **AB** into five equal parts. Draw any line, **AX** at a convenient angle. On **AX** mark off five equal lengths, **AC**, **CD**, **DE**, **EF**, **FG**. Join **BG**; and through the points **C**, **D**, **E**, **F** draw lines parallel to **BG**, viz. **CP**, **DQ**, **ER**, **FS**. Then the points **P**, **Q**, **R**, **S** will divide **AB** into five equal parts.

For by Theorem I. the segments of **AB** are proportional to the segments of AG. But the segments of AG are all equal; therefore the segments of AB are all equal.

Problem V.—To find the mean proportional to two given lengths, p, q.

Take any line AX, and on it mark off AB $= p$, BC $= q$. Describe a semicircle on diameter AC. From B draw a line perpendicular to AC, and meeting the circumference at the point D. Then BD will be the required mean proportional; *i.e.*—

$$p : \text{BD} = \text{BD} : q$$

Theorem II.—If two lines are cut by a series of parallel lines, the intercepts on the one line shall be proportional to the intercepts on the other. (See Fig. 126.)

PQ is cut at the points ABCD by a series of parallel lines; the "intercepts" are the portions intercepted between the successive parallels, viz. AB, BC, CD. RS is cut by the same series of parallel lines, which form the intercepts EF, FG, GH. Then—

$$\text{AB} : \text{BC} : \text{CD} = \text{EF} : \text{FG} : \text{GH}$$

To prove this, join DE, cutting BF and CG at L and M respectively. Then applying Theorem I. to the triangle ADE, we see that—

$$\text{AB} : \text{BC} : \text{CD} = \text{EL} : \text{LM} : \text{MD}$$

Also, applying Theorem I. to the triangle EDH, we see that—

$$\text{EL} : \text{LM} : \text{MD} = \text{EF} : \text{FG} : \text{GH}$$

It follows that—

$$\text{AB} : \text{BC} : \text{CD} : = \text{EF} : \text{FG} : \text{GH}$$

Theorem III.—The line which bisects the angle between two sides of a triangle will divide the third side proportionally to these two sides. (See Fig. 127.)

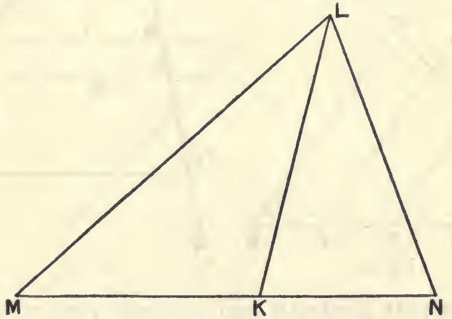


FIG. 127.

Here LK bisects the angle LMN. It will be found by actual measurement that—

$$\text{MK} : \text{KN} = \text{ML} : \text{LN}$$

EXAMPLES.—CXXXI.

1. Draw a triangle whose sides are 2, 2·5, and 3 inches in length. Bisect the angle between the two shorter sides by a line cutting the third side. Measure the segments of the third side; and show by calculation that the ratio of the segments of the third side is equal to the ratio of the other two sides.

2. Draw a triangle whose sides are 3, 3·5, 2·5 inches in length. Divide the shortest side into two portions of lengths ·75, 1·75 respectively. Through the point of division draw a line parallel to the longest side and cutting the third side. Measure the segments of the third side, and show by calculation that they are in the same ratio as the segments of the shortest side.

3. Find by drawing and measure the fourth proportional to the lengths: (i.) ·56, ·37, 1·24 inches; (ii.) 1·3, 2·12, ·84 inches; (iii.) 1·25, ·85, 1·72 inches.

4. Find by drawing and measure the third proportional to the lengths: (i.) 2·2, 1·4; (ii.) 1·8, ·96; (iii.) ·88, 1·32.

5. Construct a triangle **ABC**, having **AB** = 2 inches, **BC** = 2·5 inches, **CA** = 1·5 inches. Bisect the angle **C** by a line meeting **AB** at **F**; bisect the angle **B** by a line meeting **AC** at **E**, and **FC** at **O**. Measure the lengths **FO, OC, AE, EC**. Hence show that **FO : OC = ½AE : EC**. Deduce the same result by the use of Theorem III.

6. Draw a line of length 3 inches, and divide it proportionally to the lengths 1·22, 2·24, ·88, ·66. Measure the segments. What is the ratio of each of the given lengths to the corresponding segment of the given line?

7. Divide a line of length 2·5 inches proportionally to the lengths ·82, ·38, 1·22, ·58. Measure the segments, and find the ratio of each of the given lengths to the corresponding segment of the given line.

8. Given that $a = 1{\cdot}44$ inches, $b = {\cdot}86$ inch, $c = 1{\cdot}28$ inches; find, by geometrical construction, a length d such that $a : b = d : c$. (Find the fourth proportional to b, a, c.)

9. Find the mean proportionals to the lengths: (i.) 2·2, ·89; (ii.) 1·8, ·51; (iii.) ·88, 1·98.

221. On Similar Triangles.

Theorem IV.—If the three angles of one triangle are respectively equal to the three angles of another triangle, the sides of the first triangle are ***proportional*** **to those of the second;** ***the corresponding sides being those which are opposite to the equals.***

Such triangles are said to be **similar** to one another.

Thus in Fig. 128 we have three triangles having equal angles: viz. angle **A** = angle **D** = angle **G**; angle **B** = angle **E** = angle **H**; angle **C** = angle **F** = angle **K**. By measuring all the sides in inches, we find **AB : BC : CA** = ·5 : ·65 : ·78 = 1 : 1·3 : 1·56; **DE : EF : FD** = ·77 : 1 : 1·2 = 1 : 1·3 : 1·56; **GH : HK : KG** = 1 : 1·3 : 1·56. Hence in each triangle the sides are proportional to 1, 1·3, 1·56.

Note that since the angle **C** is equal to the angle **K**, the sides opposite to these angles are corresponding sides—that is, the side **AB** in the triangle **ABC** corresponds to the side **GH** in the triangle **GHK.**

It is obvious that **similar triangles are of the** *same shape* **but** of *different size.* The triangle **GHK** is of the same shape as the other two triangles, but is in a reversed position, the left-hand side of this triangle corresponding to the right-hand side in the others.

Note that in order to show that two triangles are similar, it is sufficient to prove that *two angles in the one triangle are respectively equal to two angles in the other.* For since the three angles of each triangle are together equal to two right angles, the third angle in the one triangle must also be equal to the third angle in the other.

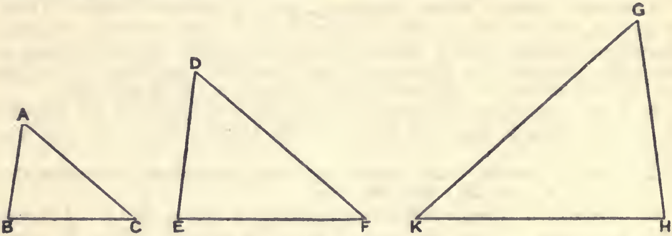


FIG. 128.

222. Theorem V.—If the sides of one triangle are proportional to the sides of another, the two triangles will be similar; the three angles of the one triangle will be equal to the three angles of the other, equal angles being opposite to corresponding sides.

Theorem VI.—If one angle of one triangle is equal to one angle of another triangle, and if the sides which contain this angle in the one triangle are proportional to the sides which contain this angle in the other triangle, the two triangles will be similar.

These two theorems are obviously variations of Theorem IV.

EXAMPLES.—CXXXII.

1. Construct a triangle whose sides measure ·75, 1, 1·25 inches respectively, and another whose sides are double those of the first. Show by measurement that the angles of the first triangle are equal to those of the second.

2. Using the same figure, measure the line joining any angular point to the middle of the opposite side in the smaller triangle, and show that it is exactly half the length of the corresponding line in the larger triangle (*i.e.* of the line joining the corresponding angular point to the middle of the opposite side).

3. Construct a triangle having an angle of 120° contained between two sides of length 1·5 and 2·5 inches respectively. Construct a second triangle having an equal angle contained between two sides 1·4 times as large as in the first triangle. Show by measurement that the two triangles are equiangular.

4. Using the same figure, show that the perpendicular from either angular point of the second triangle to the opposite side (produced, if necessary) is 1·4 times as large as the corresponding perpendicular in the first triangle.

5. In the triangle ABC, AB = ·25 inch, AC = ·5 inch, angle BAC = 60°. Draw a triangle "**of four times the linear dimensions**" (*i.e.* having each *length* four times the given length, but the *angle* unchanged); measure the third size and the largest angle. [Result, 1·73; 90°.

6. The sides of a triangle measure 6·93, 6·93, and 12 inches respectively. Construct a triangle of one-fifth the linear dimensions; measure the obtuse angle, and also the length of the perpendicular drawn from either acute angle to the opposite side produced. Deduce the magnitude of the corresponding angle and perpendicular in the original triangle. [Result, 120°, 1·2; 120°, 6.

7. Construct a triangle on a base 2 inches, having the angles at the base equal to 60° and 70° respectively. Construct a second triangle on a base 1·5 inches, having the same base angles as the first. Show by measurement that each side of the second triangle is three-fourths of the corresponding side in the first triangle, and that the altitude of the second triangle is three-fourths of the altitude of the first.

223. Any two figures which are of the same shape but of different size are called **similar figures.** Thus in Fig. 129 the two polygons are similar.

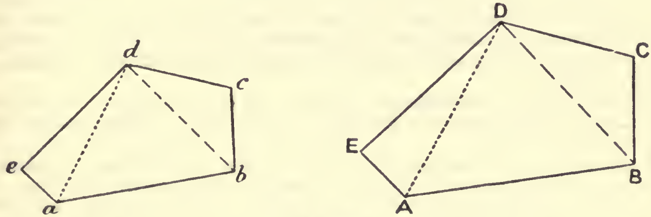


FIG. 129.

Any angle in the one polygon is equal to the corresponding angle in the other; *e.g.* angle *cda* = angle CDA, angle *dae* = angle DAE. The ratio of any two lengths in the one polygon is equal to the ratio of the corresponding lengths in the other polygon; *e.g.* *ad* : *bc* = AD : BC.

Theorem VII.—In similar figures corresponding angles are equal, corresponding lengths are proportional.

In order to test the similarity of two polygons, it is necessary to divide them up into corresponding triangles, as in Fig. 129. If the triangles of the one polygon are respectively similar to those of the other, the complete polygons are also similar.

It is very important for the student to notice that the tests for similarity of triangles cannot be directly extended to polygons. For example, if the three angles of one triangle are respectively equal to the three angles of another, the triangles are similar (Theorem IV.); but if the four angles of a quadrilateral are respectively equal to the four angles of another, the quadrilaterals *are not necessarily similar.* Thus the four angles of a square are respectively equal to the four angles of a rectangle; but a square is not similar to a rectangle, for in the one case the sides are all equal, in the other only opposite sides are equal.

Theorem VIII.—Two polygons are similar, provided that it is possible to divide them into triangles in such a way that

each triangle in the one polygon is similar to the corresponding triangle in the other.

Problem VI.—To describe a polygon similar to a given polygon *abcde*, being given the side AB of the new polygon which is to correspond to the side *ab*. (See Fig. 129.)

Divide *abcde* into triangles. At A make the angle BAD equal to the angle *bad;* at B make the angle ABD equal to the angle *abd*. Then the triangle ABD is similar to the triangle *abd* (Theorem IV.). In the same way describe the triangles DAE, DBC similar to the triangles *dae*, *dbc*.

Examples.—CXXXIII.

1. Construct the quadrilateral of Question 1, Examples CVI. Construct also a similar quadrilateral *abcd*, in which *ab* measures 1·5 inches. Measure *bd*.

2. Construct the quadrilateral of Question 3, Examples CVI. Construct also a similar quadrilateral *hklm*, to be 1·5 times as great in linear dimensions. Measure *hl* and *km*.

3. Construct the quadrilateral of Question 4, Examples CVI. Construct also a similar quadrilateral *abcd*, having $bc = 2·7$ inches. Measure *ac* and the angle *dab*.

4. Construct a quadrilateral *hklm* similar to that of Question 7, Examples CVI., to be ·6 times as great in linear dimensions. Measure the angle *lmh*.

5. Take a circle of radius 1·4 inches; divide its circumference into five equal arcs, by trial; joining the points of division form the regular pentagon ABCDE; join AD, CE, intersecting at G; draw BF perpendicular to CE.

Also in a circle of radius 1 inch describe a regular pentagon *abcde;* join *ad*, *ce*, intersecting at *g;* draw *bf* perpendicular to *ce*.

Prove by measurement that angle EGD = angle *egd;* angle BAD = angle *bad;* $AB : ab = AD : ad = BF : bf = AG : ag$.

224. On the Use of Accurate Diagrams.—An accurate plan of a field would be a figure geometrically similar to the field itself. Every length on the plan should be proportional to the corresponding length on the field; every angle on the plan would be equal to the corresponding angle on the field. Thus, provided we know the "**scale of representation,**" we can infer all particulars with regard to shape and size of the field from its plan.

The same considerations apply to all diagrams of plane figures. The following examples will indicate how to solve easy problems by their use:—

Example (1).—*There are two consecutive milestones* A *and* B *on a straight road. A church-spire* S *can be seen across the fields. On measuring* the angles* BAS *and* ABS, *we find them to be* 70° *and* 50° *respectively. Calculate in yards the distance of the spire from the nearest point of the road.*

First draw a rough diagram, as in Fig. 130. The nearest point of the road

---

* A surveyor would measure such angles by means of a theodolite, which is essentially a telescope mounted on an angular scale.

to the spire will be **K**, the foot of the perpendicular from **S**. From this diagram we can easily see how to construct an accurate diagram.

AB is to represent 1760 yards. Hence, if we choose 1000 yards to an inch as our "**scale of representation**," we must make **AB** exactly 1·76 inches long. Constructing the angle 70° at **A**, and the angle 50° at **B**, we obtain the accurate position of **S** on the diagram. Draw **SK** perpendicular to **AB**; **SK** will be found to measure 1·46 inches. Thus **SK** represents a distance of 1·46 × 1000 yards = 1460 yards.

EXAMPLE (2).—*A ladder* 30 *feet long is placed against a wall with its foot at a distance of* 11·7 *feet from the foot of the wall. Find the slope of the ladder to the horizontal, and also to what height the ladder reaches on the wall.*

Fig. 131 is the rough diagram; **AC** represents the ladder. For the accurate diagram a convenient scale of representation will be 10 feet to the inch. This makes **AB** = 1·17 inches, **AC** = 3 inches.

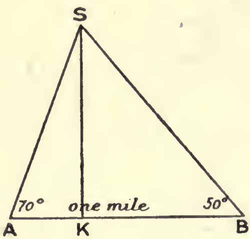


FIG. 130.

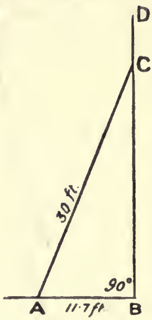


FIG. 131.

To construct the accurate diagram, draw **AB** = 1·17 inches; draw **BD** perpendicular to **AB**; with centre **A**, and radius 3 inches, describe an arc cutting **BD** in **C**; join **AC**. Measuring the angle **CAB**, we find that the ladder slopes at 67° to the horizontal. **BC** is found to be 2·76 inches, which represents a length of 10 × 2·76 feet, *i.e.* 27·6 feet.

EXAMPLES.—CXXXIV.

1. **P, Q, R** are three trees. The distance of **P** from **Q** is 117 yards, and of **P** from **R** is 76 yards; the angle **QPR** is 50°: find the distance of **Q** from **R**.

2. **P, Q** are two posts on one bank of a river; **R** is a post on the opposite bank. If **PQ** = 163 yards, angles **PQR** = 90°, **QPR** = 30°: find the breadth of the river.

3. Find the answer to the preceding problem, given the measurements, **PQ** = 156 yards, angles **PQR** = 48°, **QPR** = 80°. (Note that the breadth must be measured at right angles to **PQ**.)

4. The sides of a triangular sheet of metal are respectively 58, 87, 116 inches respectively: find its angles.

5. **AB** is a flagstaff; **C** is a point on the ground 200 feet from its foot; angle **BCA** = 22°: find the height of the flagstaff.

6. A is a balloon; C, D are two points on the ground, both due north of the balloon; DC = 1470 feet; angles ADC = 32°, ACD = 138°: find the height of the balloon above the ground.

7. A and B are two ships at sea; C and D are two coast-guard stations on the shore, 10 miles apart; angles ACD = 120°, BCD = 60°, ADC = 30°, BDC = 60°: find the distance between the ships.

8. ABCD is a rectangular field; E is a point in BC; AB = 220 yards, BC = 528 yards, BE = 165 yards. Determine, by a diagram, what distance a man saves by walking direct from A to C, instead of walking from A to E and from E to C.

225. Theorem IX.—The areas of similar triangles are in the ratio of the squares of corresponding sides.

Thus if one triangle is of three times the linear dimensions of another, its area will be 3^2 times the area of the other. For the area

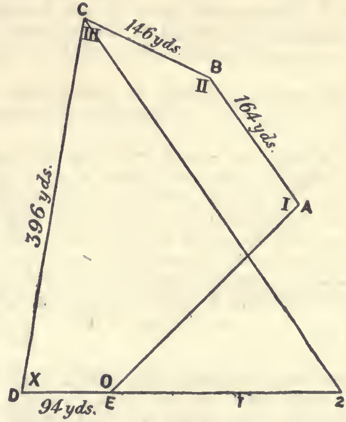


FIG. 132.

of each triangle is half the product of its base and altitude; also the base and altitude of the first triangle will be each three times the base and altitude of the second triangle (Theorem VII.). Thus the area of the first triangle will be **9** times that of the second.

Theorem X.—The areas of any two similar figures are in the ratio of the squares of corresponding lengths.

For example, if one figure is of five times the linear dimensions of another, then every triangle in the first figure will be of five times the linear dimensions of the corresponding triangle in the second figure. Hence, by Theorem IX., the **area** of every triangle in the first figure will be **25** times that of the corresponding triangle in the second figure; hence the whole area of the first figure will be 25 times the whole area of the second figure.

In the same way, if we are using accurate diagrams to determine an area, and if the scale of representation is 100 feet to 3 inches, then the scale of representation of area will be not 100 sq. feet to 3 sq. inches, but

$$100^2 \text{ sq. feet to } 3^2 \text{ sq. inches}$$

EXAMPLE.—*Determine from a diagram the area of a rectilinear field* ABCDE, *from the following data:* AB = 164 *yards, angle* ABC = 150°, BC = 146 *yards, angle* BCD = 75°, CD = 396 *yards, angle* CDE = 80°, DE = 94 *yards.*

For this figure a convenient scale of representation will be 200 yards to the inch, then 164 yards will be represented by ·82 inch, 146 yards by ·73 inch, etc. *The angles of the diagram will be the same as the angles of the field.* Thus the diagram in Fig. 132 represents the field. This reduces to the equivalent triangle **X 2 III**, whose area is $\frac{1}{2} \times 1·98 \times 1·66$ sq. inches, *i.e.* 1·64 sq. inches. But since 1 *inch* on the diagram represents 200 yards, 1 *sq. inch* represents *a square of land measuring* 200 *yards each way; i.e.* 1 sq. inch represents 40,000 sq. yards; hence the area of the diagram, viz. 1·64 sq. inches, represents 1·64 × 40,000 sq. yards, *i.e.* 65,600 sq. yards, or 13·55 acres.

EXAMPLES.—CXXXV.

1. The three sides of a triangular field measure respectively 200, 300, and 400 yards. Draw a diagram of the field to any convenient scale, and determine its area in square yards.

2. The three sides of a triangular field measure respectively 2430, 3570, and 4200 yards: draw a diagram of the field to any convenient scale, and determine its area.

3. A quadrilateral sheet of iron **ABCD** has the following measurements: **AB** = 20·3 inches, **BC** = 35·2 inches, **CD** = 25·7 inches, angle **ABC** = 90°, angle **BCD** = 80°: draw a diagram to scale, and determine the area.

4. Find, by means of a diagram drawn to scale, the area of a quadrilateral field **HKLM**, given that **HK** = **LM** = 212 yards, **HM** = 87 yards, angle **KHM** = 120°, angle **HML** = 130°.

5. Find, by means of a diagram drawn to scale, the area of a plot of ground **PQRS**, given that **PQ** = 32·2 yards, **QR** = 64 yards, **RS** = 43·6 yards, **SP** = 33·4 yards, **PR** = 50 yards.

6. Find, by means of a diagram drawn to scale, the area of the smaller segment cut off from a circle of radius 257 metres by a chord of length 400 metres.

CHAPTER XXV.

ON THE TRIGONOMETRICAL RATIOS.

226. The Use of Trigonometry.—The science of **Trigonometry** deals, in the first place, with the relations between the sides and angles of a triangle; by means of the properties of triangles, by an easy extension, Trigonometry deals with any kind of rectilinear figures.

227. The Measurement of Angles.—We have already explained, in Chapter XV., the method of measuring angles in degrees, minutes, and seconds. There is, however, another system of measuring angles with which the student should become acquainted.

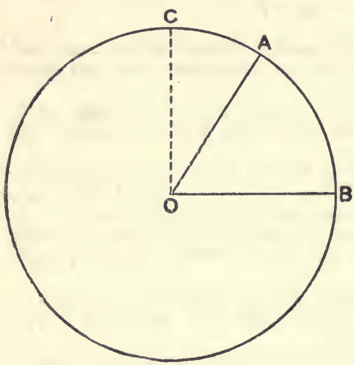


FIG. 133.

This system of measuring angles is called **Circular Measure**; it is of considerable importance, because many advanced Trigonometrical formulæ are much more conveniently expressed by its use.

The unit employed in Circular Measure is *the angle subtended at the centre of any circle by an arc equal to its radius.* Such an angle is called a **Radian.**

Thus in Fig. 133, if the length of the arc **AB** is equal to the radius **AO**, then the angle **AOB** is a radian.

Now, it can be shown that *the magnitude of a radian does not depend on the size of the circle from which it is obtained.* To prove this, draw **OC** perpendicular to **OB**; then the length of the arc **BC** is $\frac{\pi}{2} \times$ **OA** (by Formula 5, § 172). But it is obvious that the angles **AOB, COB** are in the same ratio as the arcs **AB, CB**; but $AB : CB = AO : \frac{\pi}{2} \times AO = 2 : \pi$. Thus $AB = \frac{2}{\pi} \times CB$; hence angle **AOB** $= \frac{2}{\pi} \times$ angle $COB = \frac{2}{\pi}$ of a right angle. Remembering that $\pi = 3{\cdot}14159$, we find that $\frac{2}{\pi}$ of a right angle $= 57{\cdot}295$ degrees.

228. To express any Angle in Circular Measure.

RULE.—**To express any angle in circular measure, express the angle in terms of a right angle; then multiply by $\frac{\pi}{2}$.**

EXPLANATION.—We have proved, in the last article, that 1 radian $= \frac{2}{\pi}$ right angles. It follows that 1 right angle $= \frac{\pi}{2}$ radians; and, therefore, to reduce right angles to radians, we must multiply by $\frac{\pi}{2}$.

EXAMPLE (1).—*Reduce* 35° 23′ 4″ *to circular measure.*

We first reduce the angle to the decimal of a right angle, and then multiply by $\frac{\pi}{2}$.

60	4
60	23·066 . . .
90	35·3844 . . .
	·39316 . . .

$$\cdot39316 \times \frac{3\cdot14159}{2} = \cdot61757$$

Thus the required angle = ·61757 of a radian.

The student must remember that, in Trigonometry, an angle is given in circular measure, *unless otherwise stated.* Thus "the angle ·56" means ·56 radians, not ·56 degrees.

229. On the Trigonometrical Ratios of Acute Angles.—To find the trigonometrical ratios of an acute angle we draw a perpendicular to one arm of the angle from any point in the other arm, thus forming a right-angled triangle. The side of this triangle opposite to the right angle is called the **hypotenuse,** the side opposite the original angle is called the **perpendicular,** and the third side is called the **base.**

The ratios of these three sides to one another are named as follows:—

The ratio	perpendicular	: hypotenuse	is called the	**sine**	of the angle
,,	base	: hypotenuse	,,	**cosine**	,,
,,	perpendicular	: base	,,	**tangent**	,,
,,	hypotenuse	: perpendicular	,,	**cosecant**	,,
,,	hypotenuse	: base	,,	**secant**	,,
,,	base	: perpendicular	,,	**cotangent**	,,

Thus if we wish to find any of the trigonometrical ratios of the angle A in Fig. 134, from any point P in one arm AB we draw a perpendicular PQ to the other arm AC. Then AP is the hypotenuse, PQ is the perpendicular, AQ is the base.

Then the sine of A is the ratio $\frac{PQ}{AP}$; the cosine of A is the ratio $\frac{AQ}{AP}$; the tangent of A is the ratio $\frac{PQ}{AQ}$, etc.

The utility of these ratios consists in the fact that they depend only on the *magnitude* of the angle A; the values of the hypotenuse, perpendicular, and base depend also on the position of the point P; but the *ratios* of these three lines *are the same wherever* P *is placed.*

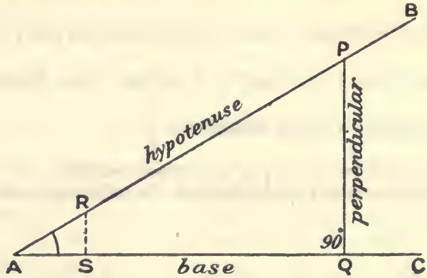


FIG. 134.

Thus we may take some other point R, and from R draw the perpendicular RS to the other arm; then using the triangle ARS, AR is the hypotenuse, RS is the perpendicular, and AS is the base. Thus the sine of A is the ratio $\frac{RS}{AR}$; the cosine of A is the ratio $\frac{AS}{AR}$, etc. But we should then find that $\frac{RS}{AR} = \frac{PQ}{AP}$; $\frac{AS}{AR} = \frac{AQ}{AP}$, etc.; and in general that we obtain the same values of the trigonometrical ratios of the angle A, whether we use the triangle APQ or the triangle ARS.

The reason for this is simply that the two triangles APQ, ARS are similar. For the angle A is common to both triangles, and the angle AQP = angle ARS, since both are right angles. It follows at once that the ratios of the sides of the one triangle are equal to the ratios of the corresponding sides of the other.

The definitions of the trigonometrical ratios are most easily remembered as follows :—

$$\text{sine} = \frac{p}{h};\ \text{cosecant} = \frac{h}{p}$$
$$\text{cosine} = \frac{b}{h};\ \text{secant} = \frac{h}{b}$$
$$\text{tangent} = \frac{p}{b};\ \text{cotangent} = \frac{b}{p}$$

where p, b, and h represent the **perpendicular, base, and hypotenuse** respectively.

It should be noted that the cosecant, secant, and cotangent are the reciprocals of the other three trigonometrical ratios.

It is usual to use the following **abbreviations** for these ratios: for sine "**sin**," for cosine "**cos**," for tangent "**tan**," for cosecant "**cosec**," for secant "**sec**," for cotangent "**cot**." Thus "**tan 40°**" means the tangent of an angle of 40°.

EXAMPLE (1).—*Determine by drawing the trigonometrical ratios of an angle of* 67°.

We draw an angle of 67°, **POQ** (see Fig. 135). We then take any point **A** in one arm, and from it draw **AB** perpendicular to **OQ**.

Measuring the sides of the triangle **OAB**, we find the hypotenuse **OA** = 2·09 inches, the perpendicular **AB** = 1·92 inches, the base **OB** = ·82 inch.

Hence $\sin 67° = \frac{1·92}{2·09} = ·92$; $\cos 67° = \frac{·82}{2·09} = ·39$; $\tan 67° = \frac{1·92}{·82} = 2·3$; $\text{cosec } 67° = \frac{2·09}{1·92} = 1·09$; $\sec 67° = \frac{2·09}{·82} = 2·55$; $\cot 67° = \frac{·82}{1·92} = ·42$.

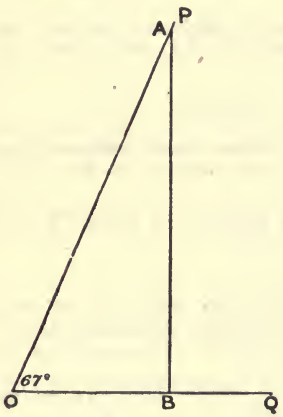


FIG. 135.

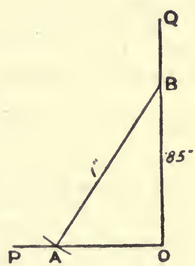


FIG. 136.

EXAMPLE (2).—*Construct the angle whose sine is* ·85, *and measure it in degrees.*

Draw a right angle POQ (Fig. 136); on OQ mark off OB = ·85 inch; with centre B, and radius 1 inch, describe an arc cutting OP at A. Join AB. BAO is the required angle.

For sin BAO = OB ÷ AB = ·85 ÷ 1 = ·85. By measuring we find that BAO = 58°.

Note that this is equivalent to the problem: Construct a right-angled triangle whose hypotenuse is 1 inch, and whose perpendicular is ·85 inch.

Note also that a larger, and therefore more accurate figure would be obtained by trebling these lengths; *i.e.* make OB = 2·55, AB = 3.

EXAMPLE (3).—*Construct the angle whose cosine is* ·57, *and measure it in degrees.*

Draw a right angle POQ (Fig. 137); mark off OA = ·57 inch, with centre A, and radius 1 inch, describe an arc cutting OQ at B. Then BAO is the required angle.

For cos BAO = ·57 ÷ 1 = ·57. By measuring we find BAO = 55°.
(We obtain a more accurate figure if we make AO = 3 × ·57, AB = 3.)

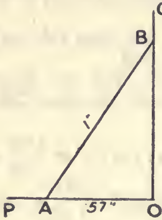


FIG. 137.

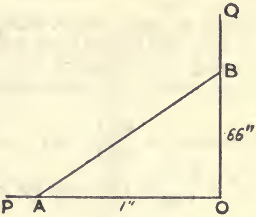


FIG. 138.

EXAMPLE (4).—*Construct and measure the angle whose tangent is ·66.*

In Fig. 138 mark off OB = ·66 inch, OA = 1 inch. Then tan BAO = ·66 ÷ 1 = ·66. BOA = 33°.

EXAMPLES.—CXXXVI.

(All angles are expressed in circular measure, unless otherwise stated.)

1. Reduce the following angles to circular measure: (i.) 30°; (ii.) ·1 right angle; (iii.) 24·5°; (iv.) 20° 12′.

2. Express the following angles in degrees: (i.) $\frac{\pi}{6}$ (ii.) $\frac{2\pi}{5}$; (iii.) 1·12; (iv.) ·587; (v.) ·035; (vi.) $\frac{3\pi}{20}$.

3. If two angles of a triangle measure one right angle and one radian respectively, express the third angle of the triangle in circular measure.

4. Determine, by drawing and measuring, the following ratios: sin 20°, cos 40°, tan 70°, sec 37°, cosec 48°, cot 35°.

5. Draw and measure the angles whose sines are: (i.) ·25; (ii.) ·14; (iii.) ·59; (iv.) ·42; (v.) ·9; (vi.) 1·2; (vii.) 1·8.

6. Draw and measure the angles whose cosines are: (i.) ·3; (ii.) ·48; (iii.) ·72; (iv.) ·8; (v.) ·18; (vi.) 1·2; (vii.) 1·6.

7. Draw and measure the angles whose tangents are: (i.) ·3; (ii.) ·5; (iii.) ·75; (iv.) 1; (v.) 1·5; (vi.) 3·5.

8. Find the values of (i.) cos $\frac{\pi}{6}$; (ii.) sin ·5; (iii.) tan 1.

230. Trigonometrical Ratios determined by Calculation.—The trigonometrical ratios of the angles 30°, 45°, 60° can be determined by elementary calculation.

Fig. 139 represents an equilateral triangle ABC of side 2 inches. (The figure is not drawn to correct size.) D is the middle point of BC; then, since the triangle is equilateral, AD is perpendicular to BC, and bisects the angle at AB; D and DC must each be 1 inch.

Now, the three angles of the triangle ABC are together equal to 180° (see Theorem VI., Chapter XV.); as they are all equal, each angle must be 60°; thus each half-angle at A must be 30°.

By using Theorem IX., Chapter XV., we can determine the length AD. For by this theorem $AB^2 = BD^2 + AD^2$; but $AB^2 = 4$, and $BD^2 = 1$; hence the value of AD^2 must be 3. Thus the length of AD must be $\sqrt{3}$ inches, *i.e.* 1·732 inches.

Using the triangle ABD, we can determine the trigonometrical ratios of the angle 60°.

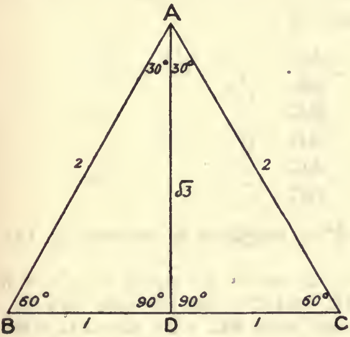


FIG. 139.

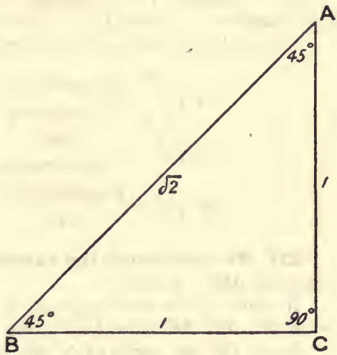


FIG. 140.

$$\text{For } \sin 60° = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{AD}{AB} = \frac{\sqrt{3}}{2} = \cdot 866$$

$$\cos 60° = \frac{\text{base}}{\text{hypotenuse}} = \frac{BD}{AB} = \tfrac{1}{2} = \cdot 5$$

$$\tan 60° = \frac{\text{perpendicular}}{\text{base}} = \frac{AD}{BD} = \frac{\sqrt{3}}{1} = 1 \cdot 732.$$

Again, from the same triangle we can determine the trigonometrical ratios of the angle 30°. For this purpose *we must regard the side opposite* 30° *as the perpendicular;* for we must suppose that from B in the arm AB of the angle 30° we have drawn the perpendicular BD to the other arm AD.*

$$\text{Thus } \sin 30° = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{BD}{AB} = \tfrac{1}{2} = \cdot 5$$

$$\cos 30° = \frac{\text{base}}{\text{hypotenuse}} = \frac{AD}{AB} = \frac{\sqrt{3}}{2} = \cdot 866$$

$$\tan 30° = \frac{\text{perpendicular}}{\text{base}} = \frac{BD}{AD} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}\dagger = \cdot 577$$

---

* In this connection it is important to note that to find the ratios of any angle, we must have a triangle containing the given angle, and also containing a right angle; the *side opposite to the right angle* counts as the *hypotenuse*, and the *side opposite to the given angle* as the *perpendicular*.

† See § 97.

Fig. 140 enables us to determine the ratios of an angle of 45° We draw AC and CB at right angles, and each equal to 1 inch, and then join AB. Then by Theorem IV., Chapter XV., the angles at A and B are equal; moreover the three angles A, B, and C are together equal to 180° (Theorem VI., Chapter XV.). Hence each of the angles A and B must be 45°.

By Theorem IX., Chapter XV., $AB^2 = AC^2 + BC^2 = 1 + 1 = 2$; hence $AB = \sqrt{2} = 1{\cdot}414$.

Thus finding the ratios of the angle B—

$$\sin 45^\circ = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = {\cdot}707$$

$$\cos 45^\circ = \frac{\text{base}}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{1}{\sqrt{2}} = {\cdot}707$$

$$\tan 45^\circ = \frac{\text{perpendicular}}{\text{base}} = \frac{AC}{BC} = \frac{1}{1} = 1.$$

231. To determine the ratios of an angle of 0° we use Fig. 141. Suppose AB = 1 inch.

In this figure the angle A is small, but is not equal to 0°; if it were, the line AC would lie along AB, and C would coincide with B.

Thus for an angle of 0° we should have BC = 0; also AC = AB = 1.

Thus $\sin 0^\circ = \frac{0}{1} = 0$; $\cos 0^\circ = \frac{1}{1} = 1$; $\tan 0^\circ = \frac{0}{1} = 0$; $\sec 0^\circ = \frac{1}{1} = 1$.

FIG. 141.

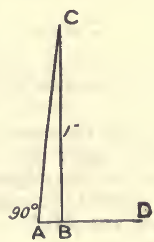


FIG. 142.

Also $\text{cosec } 0^\circ = \frac{1}{0}$; but when the denominator of a fraction is very much smaller than the numerator, the fraction becomes very great; thus when the denominator becomes indefinitely small, the fraction becomes indefinitely great, or (as we usually express it) becomes equal to infinity. The symbol for infinity is ∞.

Thus $\text{cosec } 0^\circ = \infty$; and similarly $\cot 0^\circ = \infty$.

To determine the ratios of an angle of 90°, we use Fig. 142. Suppose BC = 1 inch.

In this figure the angle A is not quite 90°; if it were, the line CB would coincide with **CA**; for **CA** and **CB** could not both be perpendicular to **AD**, unless they were coincident.

Thus for an angle of 90° we should have **BC** = 0; also **CA** = **CB** = 1.

Thus $\sin 90° = \frac{1}{1} = 1$; $\cos 90° = \frac{0}{1} = 0$; $\tan 90° = \frac{1}{0} = \infty$; $\operatorname{cosec} 90° = \frac{1}{1} = 1$; $\sec 90° = \frac{1}{0} = \infty$; $\cot 90° = \frac{0}{1} = 0$.

The following table contains the results of the last two paragraphs:—

angle	0°	30°	45°	60°	90°
sine	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
cosine	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$
tangent	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{2}}$	$\sqrt{\frac{3}{1}}$	$\sqrt{\frac{4}{0}}$

The results are tabulated in this form to give the sequence $\sqrt{0}$, $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, which is a great aid to the memory.

From this table $\sin 0° = \frac{\sqrt{0}}{2} = 0$; $\cos 60° = \frac{\sqrt{1}}{2} = \cdot 5$; $\tan 90° = \sqrt{\frac{4}{0}} = \infty$, etc.

232. On the Table of Trigonometrical Ratios.—In a table at the end of this book will be found the values of the trigonometrical ratios of any angle between 0° and 90° containing an exact number of degrees. In this table the words at the head of each column refer to the angles quoted in the left-hand column, while the words at the foot of each column refer to the angles quoted in the right-hand column.

Thus, to find tan 27°, we find the angle 27° in the *left-hand* column, and we therefore look for the value of the tangent in the column which is marked "*tangent*" *at the top*, i.e. in the fourth column. Thus tan 27° = ·5095. To find cos 71°, we find the angle 71° in the *right-hand* column, and we therefore look for the value of the cosine in the column which is marked "*cosine*" *at the bottom*, i.e. in the third column from the left. Thus cos 71° = ·3256.

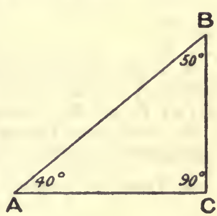


FIG. 143.

It is obvious, from the arrangement of the table, that many of the numbers registered have two meanings. For example, we see from the table that sin 40° = ·6428, which is also the value of cos 50°. From Fig. 143 it is easy to see why sin 40° and cos 50° have the same value. For if the angles **A** and **C** are made equal to 40° and 90° respectively, then the angle **B** will be 50° by Theorem VI., Chapter XV.;

this triangle may therefore be used to determine the ratios of both 40° and 50°; but if we are determining the ratios of 40°, we must reckon **BC** as the perpendicular and **AC** as the base; while if we are using it to determine the ratio of 50°, we must reckon **AC** as the perpendicular and **BC** as the base; thus *sin* 40° *and cos* 50° *are both represented by the ratio* $\frac{BC}{AB}$. A similar argument will be found to hold good for all other quantities registered in the table which have two meanings.

233. On Powers of Ratios and Inverse Functions.—The square of cos A should strictly be written "$(\cos A)^2$;" but it is usual always to represent it by the notation "$\cos^2 A$;" similarly "$\tan^2 35°$" means "the square of tan 35°," etc.

The symbol "**arcsin** x" means "the angle whose sine is x;" and similarly for the other trigonometrical ratios. Thus arctan 2 = the angle whose tangent is 2 = 63° approximately (cf. the tables).

The same quantities are often expressed by the notation $\sin^{-1} x$, $\tan^{-1} 2$, etc. Thus $\sec^{-1} 2{\cdot}5$ = arcsec $2{\cdot}5$ = the angle whose secant is $2{\cdot}5$ = 66° approximately.

The functions arcsin x, $\sin^{-1} x$, arctan x, $\tan^{-1} x$, etc., are called **inverse trigonometrical functions.**

Note carefully that there is some danger of confusion with the notation $\sin^{-1} x$. Since "$\sin^2 x$" means "$(\sin x)^2$," in the same way "*$\sin^{-1} x$" would naturally mean* "$(\sin x)^{-1}$," i.e. "$\frac{1}{\sin x}$" (see § 102); but the symbol "$\sin^{-1} x$" is *never used in this sense*, but always as meaning the angle whose sine is x.

Examples.—CXXXVII.

(All angles are expressed in circular measure, unless otherwise stated.)

1. Find the values of the following trigonometrical ratios, by reducing the angles to degrees, and using the tables: $\sin\frac{\pi}{6}$, $\cos\frac{\pi}{3}$, $\tan\frac{\pi}{4}$, $\cot\frac{\pi}{9}$, $\sec\frac{\pi}{10}$, $\operatorname{cosec}\frac{\pi}{5}$.

2. Prove the following statements, using only the definitions of the trigonometrical ratios: (i.) $\cos A . \tan A = \sin A$; (ii.) $\sin A . \cot A = \cos A$; (iii.) $\sec A . \cot A = \operatorname{cosec} A$; (iv.) $\sec A = \tan A . \operatorname{cosec} A$.

3. Prove from Theorem IX., Chapter XV., and using the definitions of the trigonometrical ratios, that $\cos^2 A + \sin^2 A = 1$.

4. Trace the graph $y = \sin x$, from $x = 0$ to $x = \frac{\pi}{2}$; and from the graph deduce the rate of increase of y with regard to x, when $x = \frac{\pi}{3}$, and when $x = \frac{\pi}{4}$. (Use the tables; let abscissæ represent radians.)

5. Trace the graph, $y = \cos x$, from $x = 0$ to $x = \frac{\pi}{2}$; and from the graph deduce the rate of variation of y with regard to x, when $x = \frac{\pi}{4}$, and when $x = \frac{\pi}{3}$.

6. Trace the graph, $y = \sin x \,.\, \cos x$, from $x = 0$ to $x = \frac{\pi}{2}$; and determine the rate of variation of y with regard to x, when $x = \frac{\pi}{6}$, and when $x = \frac{\pi}{3}$.

7. Trace the graph, $y = \cos^2 x - \sin^2 x$, from $x = 0$ to $x = \frac{\pi}{4}$; and determine the rate of variation of y with regard to x, when $x = \frac{\pi}{6}$.

8. Trace the graph, $y = \tan x + \cot x$, from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$; and determine the rate of variation of y with regard to x, when $x = \frac{\pi}{4}$.

9. From the tables at the end of the book write down the value of: (i.) $\sin^{-1} \cdot 515$, $\cos^{-1} \cdot 891$, $\tan^{-1} \cdot 364$; (ii.) $\cot^{-1} 1 \cdot 804$, $\cos^{-1} \cdot 5$, $\sin^{-1} \cdot 891$; (iii.) $\cot^{-1} \cdot 364$, $\cos^{-1} \cdot 682$, $\cot^{-1} 1 \cdot 327$, $\tan^{-1} 1$.

10. Trace the graph, $y = \tan^{-1} x$, from $x = 0$ to $x = 4$; and find the rate of variation of y with regard to x when $x = 2$.

11. Trace the graph, $y = \cos^{-1} \frac{x}{4}$, from $x = 0$ to $x = 3$, and find the rate of variation of y with regard to x, when $x = 2$.

CHAPTER XXVI.

ON THE SOLUTION OF RIGHT-ANGLED TRIANGLES, THE EVALUATION OF TRIGONOMETRIC FORMULÆ, AND POLAR CO-ORDINATES.

234. On the Solution of Right-angled Triangles.—Any triangle has three sides, and three angles. These are called the six "**parts**" of the triangle. If we know any three parts of a triangle, *including at least one side*, we can usually determine the other three parts by the application of certain formulæ. The process of determining some parts of a triangle from the others is called "**solving the triangle.**"

We have already learnt how to solve triangles by accurate drawing and measurement.

For the construction of formulæ bearing on triangles, we represent the three angles of a triangle by the capital letters A, B, C, and the

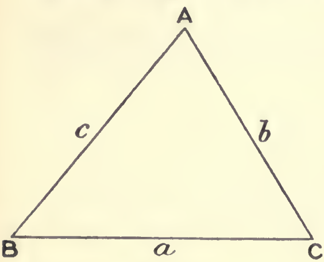


Fig. 144.

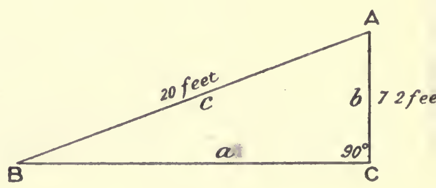


Fig. 145.

three sides by the small letters *a*, *b*, *c*. The letter *a* denotes the *side opposite to the angle* A; and similarly for *b* and *c*. (See Fig. 144.)

The solution of a right-angled triangle by Trigonometry is comparatively simple.

Case I.—Given the lengths of two sides.

RULE. — **First determine the third side by means of Theorem IX., Chapter XV.; then determine one angle from one of its trigonometrical ratios; the third angle is the complement.**

(For the meaning of the word "**complement,**" see § 147.)

EXAMPLE (1).—*In a right-angled triangle the hypotenuse is* 20 *feet, and one side is* 7·2 *feet: find the other side and angles.*

Use Fig. 145. We are given that $c = 20$, $b = 7·2$. By Theorem IX., Chapter XV., $c^2 = a^2 + b^2$; *i.e.* $20^2 = a^2 + 7·2^2$; thus $a^2 = 20^2 - 7·2^2$; whence $a = 18·66$.

To find the angle **B** :—$\sin \mathsf{B} = \frac{\mathsf{AC}}{\mathsf{AB}} = \frac{7·2}{20} = ·36$; we then find (from the table of trigonometrical ratios) the angle whose sine is ·36; the table gives $\sin 21° = ·3584$; thus **B** is a little more than 21°.

Since $\mathsf{A} + \mathsf{B} + \mathsf{C} = 180°$ (Theorem VI., Chapter XV.), and $\mathsf{C} = 90°$; therefore $\mathsf{A} + \mathsf{B} = 90°$; thus **A** is the complement of **B**; *i.e.* $\mathsf{A} = 90° - \mathsf{B} = 90° - 21° = 69°$. To be more accurate, **A** is a little less that 69°, since **B** is a little more than 21°.

Case II.—Given one side and one angle in addition to the right angle.

RULE.—**Determine the other sides from the trigonometrical ratios; and the third angle from Theorem VI., Chapter XV.**

EXAMPLE (2).—*In a right-angled triangle one angle is* 32°, *and the side opposite to this angle is* 13 *feet. Solve the triangle.*

Use Fig. 146. We are given $\mathsf{B} = 32°$, $b = 13$ feet. To find c; $\frac{b}{c} = \sin \mathsf{B} = \sin 32° = ·5299$; *i.e.* $\frac{13}{c} = ·5299$; whence $c = 13 \div ·5299 = 24·53$ feet.

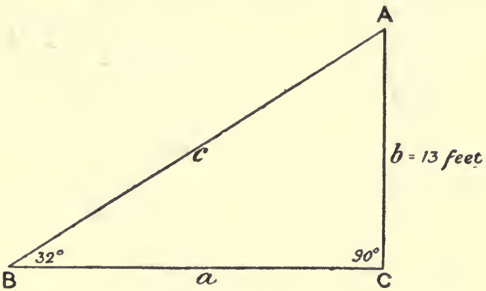


FIG. 146.

To find a; $\frac{a}{b} = \cot \mathsf{B} = \cot 32° = 1·6003$; *i.e.* $\frac{a}{13} = 1·6003$; $\therefore a = 13 \times 1·6003 = 20·8$ feet.

To find **A**; $\mathsf{A} + \mathsf{B} = 90°$; $\therefore \mathsf{A} = 90° - 32° = 58°$.

Note carefully the method by which we determine one side, knowing another side; the *ratio of the required side to the known side is one of the trigonometrical ratios of the known angle*, and therefore can be found from the table; this always enables us to determine the required side.

235. Definition.—If there are two points, **A** and **B**, and if **A** is on a higher level than **B**, then the angle between the line **AB** and the horizontal plane through **B** is called the angle of elevation of **A** at **B**. If the point **A** is on a lower level than **B**, then the angle between the line **AB** and the horizontal plane through **B** is called the angle of depression of **A** at **B**.

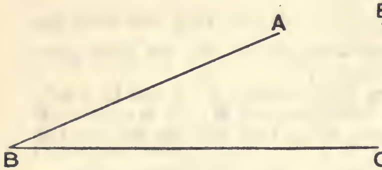


Fig. 147.

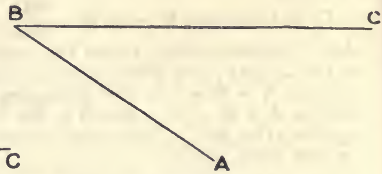


Fig. 148.

Thus in Fig. 147 the angle of elevation of **A** at **B** is the angle **ABC**; and in Fig. 148 the angle of depression of **A** at **B** is the angle **ABC**.

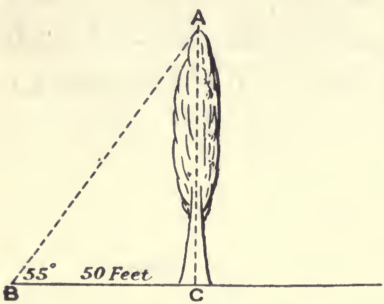


Fig. 149.

Example.—*Find the height of a tree if the angle of elevation of the top of the tree is found to be 55° at a point whose distance from the foot of the tree is 50 feet.*

Using Fig. 149, in the right-angled triangle **ABC** we are given the side **BC** = 50 feet, and the angle **B** = 55°.

To find **AC**—

$$\frac{AC}{BC} = \tan B = \tan 55° = 1{\cdot}4281;$$

i.e. $\frac{AC}{50} = 1{\cdot}4281$; $\therefore AC = 50 \times 1{\cdot}4281 = 71{\cdot}4$ feet.

Examples.—CXXXVIII.

1. **ABC** is a triangle, right-angled at **C**: find the other parts of the triangle in each of the following cases: (i.) $a = 20, c = 50$; (ii.) $a = 5, c = 8$; (iii.) $b = 170, c = 300$; (iv.) $a = 30, b = 20$; (v.) $a = 25, b = 35$.

2. **ABC** is a triangle, and $C = 90°$: find the other parts of the triangle given: (i.) $A = 25°, a = 15$; (ii.) $A = 35°, b = 25$; (iii.) $A = 72°, a = 120$; (iv.) $B = 58°, a = 100$; (v.) $B = 50°, b = 10$.

3. Find the height of a tree, given that at a distance of 30 feet from the foot of the trunk, the elevation of the top of the tree is 52°.

4. Find the height of the top of a church steeple above the ground, if at a distance of 1000 yards the elevation of the top of the steeple is 2°.

5. A flagstaff is on the top of a tower; at a distance of 1000 feet from the foot of the tower the elevation of the top of the tower is 3°, and of the top of the flagstaff is 4°. Determine the height of the tower and of the flagstaff.

6. A man at the top of a cliff 600 feet high sees two boats at sea, both due south of him; the angle of depression of the nearer boat is 14°, and of the further boat is 8°: find the distance between the boats.

7. A man is looking from a window 20 feet above the ground at a flagstaff; the angle of depression of the foot of the flagstaff is 20°, and the angle of elevation of the top of the flagstaff is 42°: find the height of the flagstaff. (Use an accurate diagram, or Trigonometry.)

8. The diagonal of a rectangular field is 530 yards long; the shorter side is inclined to the diagonal at an angle of 52°: find the length of the longer side of the field.

9. Two steamers start at the same time from the same port; the first travels at 12 miles an hour due N., and the second at 16 miles an hour due W.: find the bearing of the first steamer as seen from the second, (i.) after one hour, (ii.) after two hours.

10. A steamer, which is travelling due E. at 10 miles an hour, sights a sailing-vessel at a distance of 20 miles in a direction 25° south of east: if the sailing-vessel is travelling due N., find what rate of travelling will cause a collision.

236. On Evaluation of Trigonometrical Formulæ.—In using logarithms for evaluating trigonometrical formulæ, it must be remembered that additions and subtractions are performed arithmetically, the use of logarithms being restricted to multiplication, division, and the evaluation of powers and roots.

EXAMPLE.—*Use the tables at the end of the book to evaluate* $\sqrt[3]{4\cos^3\theta - 3\cos\theta}$ *when* $\theta = \frac{\pi}{5}$.

From the tables we find—

$$\cos\theta = \cos\frac{\pi}{5} = \cos 36° = \cdot 809$$

$$\therefore 4\cos^3\theta = 4 \times \cdot 809^3$$

$$\text{but } \log(4 \times \cdot 809^3) = \log 4 + \log(\cdot 809^3)$$

$$= \log 4 + 3\log \cdot 809 = \cdot 6021 + 3(\bar{1}\cdot 9079)$$

$$= \cdot 6021 + \bar{1}\cdot 7237 = \cdot 3258$$

$\cdot 3258$ is the logarithm of $2\cdot 117$. Thus $4\cos^3\theta = 2\cdot 117$.

$$\text{Again } 3\cos\theta = 3 \times \cdot 809 = 2\cdot 427$$

$$\text{Thus } 4\cos^3\theta - 3\cos\theta = 2\cdot 117 - 2\cdot 427 = -\cdot 31$$

$$\therefore \sqrt[3]{4\cos^3\theta - 3\cos\theta} = \sqrt[3]{-\cdot 31} = -\cdot 31^{\frac{1}{3}}$$

$$\text{but } \log(\cdot 31)^{\frac{1}{3}} = \tfrac{1}{3}\log \cdot 31 = \tfrac{1}{3}(\bar{1}\cdot 4914) = \bar{1}\cdot 8305.$$

$$\text{thus } (\cdot 31)^{\frac{1}{3}} = \cdot 6769 \qquad \text{Result}: -\cdot 6769$$

EXAMPLES.—CXXXIX.

Evaluate the following:—

1. $\frac{1}{2}bc\sin A$; (i.) when $b = 23$, $c = 59\cdot 7$, $A = 68°$; (ii.) when $b = 103$, $c = 315$, $A = 55°$.

2. $\sqrt{a^2 + b^2 - 2ab\cos C}$; when $a = 231$, $b = 357$, $C = 55°$.

3. $a\sin B\sin C\operatorname{cosec} A$; (i.) when $a = 253$, $A = 28°$, $B = 86°$, $C = 58°$; (ii.) when $a = \cdot 0004572$, $A = 48°$, $B = 79°$, $C = 53°$.

4. Sin A cos B + cos A sin B; (i.) when $A = 57°$, $B = 75°$; (ii.) when $A = 53°$, $B = 23°$.

5. $\left(\dfrac{\tan A - \tan B}{1 + \tan A \tan B}\right)^{\frac{1}{2}}$; (i.) when $A = 75°$, $B = 25°$; (ii.) when $A = 32°$, $B = 24°$.

6. Trace the graph $y = \sin \cdot 5x$, from $x = 0$ to $x = \pi$; and find the rate of change of y with regard to x, when $x = \dfrac{3\pi}{8}$, and when $x = \dfrac{7\pi}{10}$.

7. By means of a graph, find a value of x between 0 and $\dfrac{\pi}{2}$, which satisfies the equation, $\cos^2 x = \sin x$.

8. Trace the graph $y = \sin \cdot 025x + \cos \cdot 05x$ from $x = 0$ to $x = \dfrac{\pi}{2}$; and find the rate of change of y with regard to x, when $x = \dfrac{\pi}{6}$, and when $x = \dfrac{\pi}{3}$.

9. By means of a graph, find a value of x between 0 and $\dfrac{\pi}{5}$, which satisfies the equation, $\sin^2 x - \cos 2 \cdot 3x = 0$.

10. By means of a graph, find a value of x between 0 and $\dfrac{\pi}{6}$, which satisfies the equation, $\log(\sin x) + \sin\left(\dfrac{x}{2} + \dfrac{\pi}{9}\right) = 0$.

237. On Polar Co-ordinates.—This is a system sometimes used in Geometry instead of Cartesian co-ordinates (§ 200).

We use an origin **O**, sometimes called the **pole** (see Fig. 150), and a single axis **OX**. We describe the position of a point **P** by quoting the length of the line **OP**, and the magnitude of the angle **POX**. **OP** is called the **radius vector,** and is represented by the letter r; the angle **POX** is called the **vectorial angle,** and is represented by the letter θ (Gr. *thēta*, equivalent to *th*). r, θ are called the **Polar co-ordinates** of the point.

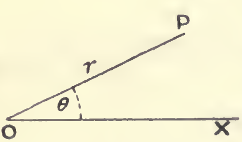


FIG. 150.

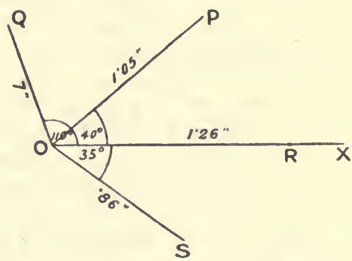


FIG. 151.

Thus in Fig. 151 the polar co-ordinates of **P** are (1·05 inches, 40°); of **Q**, are (·7″, 110°). The polar co-ordinates of **R** are (1·26″, 0°), for the angle **ROX** is an angle of 0°. The polar co-ordinates of **S** are (·86″, −35°); as the vectorial angle is drawn below **OX** instead of above it, *it is reckoned negative.*

To represent a *negative radius vector*, we take the radius vector given by the vectorial angle, and produce it backwards through **O**. Thus to represent, on Fig. 151, the point (−2 inches, 40°), we draw **OP**, which gives the correct vectorial angle; then we produce this line backwards through **O** to a point 2 inches from **O**.

A little reflection will show the student that the point (−2 inches, 40°) coincides with the point (+2 inches, −140°). Also that the point (−·7 inch, −70°) coincides with the point **Q** on Fig. 151.

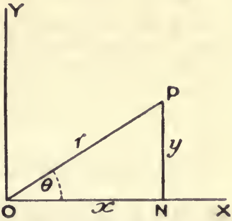


FIG. 152.

In Fig. 152, (r, θ) are the polar co-ordinates of **P**, and (x, y) are the Cartesian co-ordinates. Using the right-angled triangle **PON**, we have—

$$\frac{x}{r} = \cos\theta \text{ (§ 229)}; \therefore x = r\cos\theta \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{y}{r} = \sin\theta \text{ (§ 229)}; \therefore y = r\sin\theta \quad . \quad . \quad . \quad . \quad (2)$$

$$r^2 = x^2 + y^2 \text{ (§ 156)}; \therefore r = (x^2 + y^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad (3)$$

$$\tan\theta = \frac{y}{x} \text{ (§ 229)}; \quad \therefore \theta = \tan^{-1}\frac{y}{x} \quad . \quad . \quad . \quad (4)$$

Formulæ (1) and (2) enable us to determine the Cartesian co-ordinates of a point from the polar co-ordinates. Formulæ (3) and (4) perform the reverse operation.

EXAMPLES.—CXL.

1. Plot the following points, marking them **P, Q, R** respectively: (1 inch, 10°); (1·15 inches, 40°); (1 inch, 70°). Measure **PQ, PR**, and the angles **ORQ, RQP**.

2. Plot the points (1 inch, 40°), **P**; and (1 inch, −80°), **Q**. Measure the length **PQ**, and its inclination to **OX**.

3. Plot the points (1·2 inches, 70°), **P**; (2·4 inches, 70°), **Q**. On **PQ** describe an equilateral triangle **PQR**. Measure the polar co-ordinates of **R**.

4. Plot the points (1·41 inches, −90°), **P**; (1 inch, −45°), **Q**. Measure the polar co-ordinates of the point where **PQ** cuts **OX**.

5. Plot the points (−2 inches, 30°), **P**; (2 inches, −30°), **Q**; (−2 inches, −90°), **R**. Find the area of the triangle **PQR**.

6. Find, by calculation, the Cartesian co-ordinates of the following points: (2, 40°); (1·5, 70°); (8, 38°); (·5, 43°).

7. Find, by calculation, the polar co-ordinates of the following points: (3·36, 4·48); (·55, 1·32); (1·5, 2·12); (1·73, 3).

8. Find, by measurement, the Cartesian co-ordinates of the points: (1·5 inches, 120°); (−1·5 inches, 150°); (2 inches, −45°); (1·2 inches, −110°).

9. Find, by measurement, the polar co-ordinates of the points: (−1, 2); (1·5, −2); (−1·73, −1); (−1·5, 3).

10. Trace the graph $r = 2\cos\theta$, from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

11. Trace the graph $r = 2/\cos\theta$, from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

12. Trace the graph $r = 2/\sin\theta$, from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$.

CHAPTER XXVII.

ON SOLID GEOMETRY AND CO-ORDINATE PLANES.

238. WE have already explained, in §§ 200 and 205, how the position of a point in a plane can be determined by reference to two lines at right angles to one another, called the co-ordinate axes.

In a similar manner, the *position of a point in space can be determined by reference to three planes at right angles to one another*, called the **Co-ordinate planes.** Note that the distance of a point from a plane is measured by the length of the line drawn from the point perpendicular to the plane (see § 166).

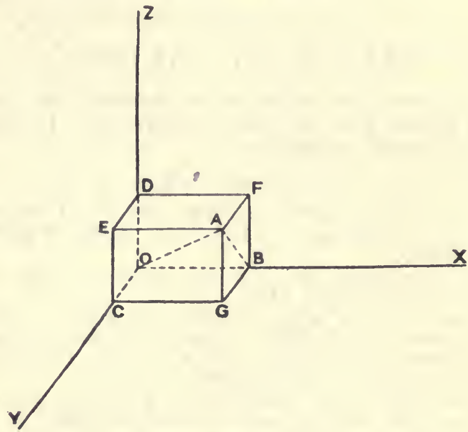


FIG. 153.

This method of determining the position of a point by reference to three planes at right angles is easily understood by imagining a box pushed into a corner of a room (see Fig. 153). Here **XOY** represents the floor, **YOZ** and **XOZ** represent two of the walls; the back of the box, **DOBF**, is touching the wall **XOZ**; the side of the box, **DOCE**, is touching the wall **YOZ**; the bottom of the box, **COBG**, is touching the floor. Now, consider the point **A**. Its distance from the wall **YOZ** is measured by the length **AE**; for **AE** is perpendicular to the side **DOCE**, and is therefore perpendicular to the wall **YOZ**. In the same way, the distance of the point **A** from the wall **ZOX** is measured by the length **AF**; and the distance of **A** from the floor **XOY** is measured by the length **AG**.

The point **O** is called the origin; the three planes **YOZ, ZOX, XOY** are called the co-ordinate planes; the lines **OX, OY, OZ** are called the co-ordinate axes (**OX** being called the axis of x, etc.); **AE, AF, AG** are the three co-ordinates of the point **A**; **AE** is called the "x" co-ordinate of **A**, because it is parallel to the axis of x; **AF** is the y co-ordinate; **AG** is the z co-ordinate.

The point whose x, y, *and* z co-ordinates are respectively 4, 3, and 2, is described as "the point (4, 3, 2)."

If a point lies in the plane **YOZ**, its x co-ordinate is 0 (since the x co-ordinate is the distance from the plane **YOZ**). Thus the co-ordinates of the point **E** are (0, **ED**, **EC**). Similarly, the co-ordinates of the point **G** are (**GC**, **GB**, 0).

If a point lies on the axis of **X** (for example, the point **B**), it lies in the plane **XOY**, and also in the plane **ZOX**; thus both the y and z co-ordinates are zero, while the x co-ordinate is **BO**.

The co-ordinates of the origin are (0, 0, 0), since it lies in all three planes.

In connection with this figure the student should carefully revise the properties of the rectangular block, § 168.

239. Formulæ in connection with Three Co-ordinate Planes—

If the co-ordinates of a point (**A**) are (x, y, z), then—

$$\mathbf{OA} = \sqrt{x^2 + y^2 + z^2} \quad \ldots \ldots \quad (1)$$

$$\left.\begin{aligned} \text{angle } \mathbf{AOX} &= \cos^{-1} \frac{x}{\sqrt{x^2 + y^2 + z^2}} \\ \text{angle } \mathbf{AOY} &= \cos^{-1} \frac{y}{\sqrt{x^2 + y^2 + z^2}} \\ \text{angle } \mathbf{AOZ} &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}\right\} \quad \ldots \quad (2)$$

Proof.—Formula (1) follows immediately from § 168, Formula (7).

To prove Formula (2), join **AB**. Then we know that **OB** is perpendicular to the plane **FBGA**; it is therefore perpendicular to the line **BA**, which is in this plane (cf. § 166); then, since the angle **ABO** is a right angle, the triangle **ABO** may be used to determine the trigonometrical ratios of the angle **AOX** (cf. § 232). Thus—

$$\cos \mathbf{AOX} = \frac{\mathbf{OB}}{\mathbf{OA}} = \frac{\mathbf{AE}}{\mathbf{OA}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

Note also that since **AGB** is a right angle—

$$\mathbf{AB} = \sqrt{\mathbf{BG}^2 + \mathbf{AG}^2} = \sqrt{\mathbf{AF}^2 + \mathbf{AG}^2} = \sqrt{y^2 + z^2}$$

$$\therefore \sin \mathbf{AOX} = \frac{\mathbf{AB}}{\mathbf{OA}} = \frac{\sqrt{y^2 + z^2}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\text{and } \tan \mathbf{AOX} = \frac{\mathbf{AB}}{\mathbf{OB}} = \frac{\sqrt{y^2 + z^2}}{x}$$

Similar formulæ can be proved in a similar manner for the angles AOY, AOZ.

240. Formulæ for the Line joining Two given Points.

RULE.—**The length of the line joining the point (x, y, z) to the point (x', y', z') is—**

$$\{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{\frac{1}{2}} \quad \ldots \ldots \quad (3)$$

the inclination of this line to the axis of x is

$$\cos^{-1} \frac{(x - x')}{\{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{\frac{1}{2}}} \quad \ldots \quad (4)$$

Note that Formulæ (3) and (4) can be obtained from Formulæ (1) and (2), by substituting $(x - x')$ for x, $(y - y')$ for y, and $(z - z')$ for z.

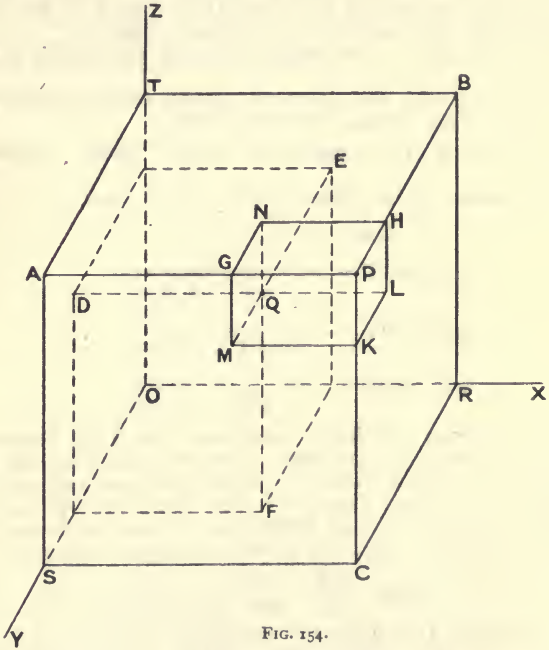


FIG. 154.

Proof.—In Fig. 154 let P be the point (x, y, z), and Q the point (x', y', z').

Drawing the rectangular block, of which P and O are the opposite corners, we see that PA $= x$, PB $= y$, PC $= z$. (Compare § 239.)

Drawing the rectangular block, of which Q and O are the opposite corners, we see that QD $= x'$, QE $= y'$, QF $= z'$.

Now, draw the rectangular block, whose opposite corners are P and Q, and whose faces are parallel to the co-ordinate planes. The face PGMK of this third block will lie in the face PASC of the first block; the face PGNH of the third block will lie in the face PATB of the first block; and the face PHLK of the third block will lie in the face PBRC of the first block. Also the edge DQ of the second block will lie in a straight line with the edge QL of the third block; the edge EQ of the second block will lie in a straight line with the edge QM of the third block; and the edge FQ of the second block will lie in a straight line with the edge QN of the third block.

Since L and P both lie in the plane PBRC, and D and A both lie in the parallel plane ATOS, and since PA and LD are parallel, it follows that PA = LD. Also PG = LQ, being parallel edges of the same rectangular block. Thus—

$$\mathrm{PG} = \mathrm{LQ} = \mathrm{LD} - \mathrm{QD} = \mathrm{PA} - \mathrm{QD} = x - x'$$

Similarly, we can show that—

$$\mathrm{PH} = \mathrm{MQ} = \mathrm{ME} - \mathrm{QE} = \mathrm{PB} - \mathrm{QE} = y - y'$$
$$\mathrm{PK} = \mathrm{NQ} = \mathrm{NF} - \mathrm{QF} = \mathrm{PC} - \mathrm{QF} = z - z'$$

Thus we see that the line PQ is the diagonal of a rectangular block whose edges measure $x - x'$, $y - y'$, $z - z'$ respectively.

$$\therefore \mathrm{PQ} = \{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{\frac{1}{2}}$$

[Formula 7, Chapter XVI.

The inclination of the line PQ to the axis of x is measured by the angle between PQ and QL (for QL is parallel to the axis of x).* Now, in § 239, Fig. 153, we have shown that the cosine of the angle between AO and OB is $\frac{\mathrm{OB}}{\mathrm{OA}}$; it follows that in this figure the cosine of the angle between PQ and QL is $\frac{\mathrm{QL}}{\mathrm{PQ}}$, *i.e.*—

$$\frac{(x - x')}{\{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{\frac{1}{2}}}$$

This proves the second formula of this paragraph. Similar formulæ hold for the inclination of PQ to the axes of y and z.

241. On Negative Co-ordinates.—If we imagine two planes at right angles to one another, such as the planes MFNC, LBHE in Fig. 155 (which intersect in the line XX′), it is obvious that these planes, if produced indefinitely in all directions, will divide the whole of space into four regions; viz.—

The first region,	bounded by the planes	LX′XB, MX′XC
„ second „	„ „ „	MX′XC, EX′XH
„ third „	„ „ „	EX′XH, FX′XN
„ fourth „	„ „ „	FX′XN, LX′XB

* Note that PQ and OX do not meet; in this case the inclination of PQ to OX is measured by the angle between PQ and any line which meets it, and which is parallel to OX.

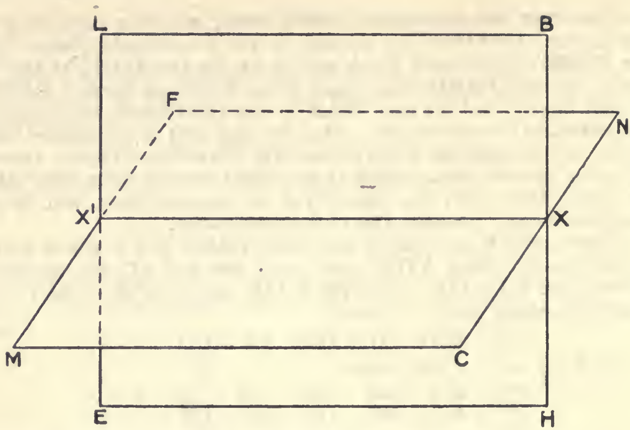


FIG. 155.

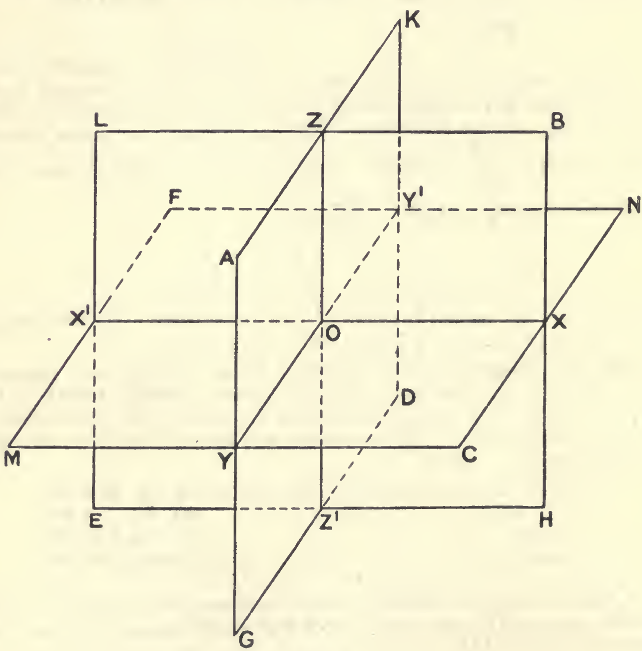


FIG. 156.

If a *third plane* be drawn at right angles to each of the first two, *e.g.* the plane AGDK in Fig. 156, each of these four regions will be again divided into two, making eight compartments in all.

If these three planes are used for the co-ordinate planes, the co-ordinates of any point will be the distances of the point from the three planes. To distinguish between the various compartments, we affix positive and negative signs to the co-ordinates in accordance with the following

RULE.—**The x co-ordinate is the distance from the plane YOZ (*i.e.* the plane AKDG), and is reckoned positive when the point lies to the *right* of this plane** (*i.e.* on the same side as the point X).

The y co-ordinate is the distance of the point from the plane ZOX (*i.e.* **the plane LBHE), and is reckoned positive when the point lies in front of this plane** (*i.e.* on the same side as the point Y).

The z co-ordinate is the distance of the point from the plane XOY (*i.e.* **the plane MFNC), and is reckoned positive when the point lies above this plane** (*i.e.* on the same side as the point Z).

It will then be found, that the *signs* of the co-ordinates determine the compartment in which the point lies; its position in that compartment is, of course, determined by the *magnitudes* of the co-ordinates.

EXAMPLE (1).—*In which compartment does the point* $(+3, -5, -2)$ *lie?*

Since the x co-ordinate is positive, the point lies to the *right* of the plane AKDG; since the y co-ordinate is negative, the point lies *behind* the plane LBHE; since the z co-ordinate is negative, the point lies *below* the plane MFNC.

Hence it must lie on the compartment bounded by the planes Y′OZ′, Z′OX, XOY′.

Formulæ (1), (2), (3), and (4) *can be applied to points in any compartment*, provided that proper account be taken of the signs of the co-ordinates.

EXAMPLE (2).—*Find the distance between the two points* $(5, -1, 7)$, $(2, 3, -5)$.

Applying Formula (3)—

$$x - x' = 5 - 2 = 3;\ \ y - y' = (-1) - 3 = -4;\ \ z - z' = 7 - (-5) = 12$$

Thus the required distance is—

$$\{(3)^2 + (-4)^2 + (12)^2\}^{\frac{1}{2}} = \{9 + 16 + 144\}^{\frac{1}{2}} = (169)^{\frac{1}{2}} = 13$$

EXAMPLE (3).—*Find the inclination of the line joining the two points* $(5, 3, 7)$, $(2, -1, -5)$ *to the axis of* y.

Formula (4) gives the inclination to the axis of x. The corresponding formula for the inclination to the axis of y is

$$\cos^{-1} \frac{(y - y')}{\{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{\frac{1}{2}}}$$

Substituting the numerical values, we obtain $\cos^{-1} \frac{4}{13} = \cos^{-1} \cdot 3077 = 72°$ approximately.

242. On Projections.—By the "**projection of a point on a plane**" we mean the foot of the perpendicular drawn from the point to the plane. Thus in Fig. 153 the projection of the point A on the plane XOY is the point G, since AG is perpendicular to XOY; the projection of the point A on the plane ZOX is the point F, since AF is perpendicular to ZOX. If the point lies in the plane, the projection on the plane is the point itself. Thus the projection of the point O on the plane XOY is the point O.

The **projection of any given figure** on a plane is the figure formed in the plane by projecting every point of the given figure.

Thus the **projection of a straight line on a plane** is the straight line formed in the plane by projecting every point in the given line. Thus in Fig. 157 the projection of the point P on the plane ABCD is the point p, and the projection of Q is the point q; then if every

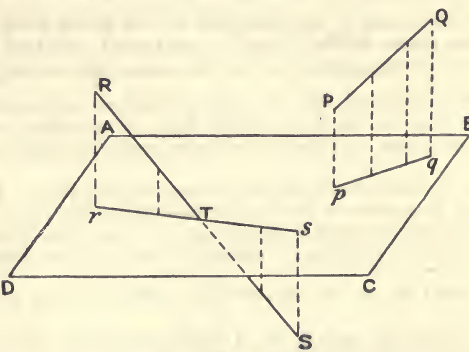


FIG. 157.

point on the straight line PQ be projected on the plane ABCD, the projections will form the straight line pq. Thus the projection of the line PQ on the plane ABCD is the straight line pq.

In the straight line RS, which cuts the plane ABCD at T, the projection of R is r, the projection of S is s, and the projection of the line RS is the line rs. Note that the projection of the line passes through the point where the line cuts the plane; *i.e.* rs passes through T.

Note that if a ship's deck is horizontal, and *the sun is vertically overhead*, then the projection on the deck of any spar or any part of the rigging will coincide with its *shadow* on the deck.

RULE.—**The angle between a straight line and a plane is measured by the angle between the line and its projection on the plane; also the length of the projection is equal to the length of the original line multiplied by the cosine of this angle.**

EXPLANATION.—In Fig. 158 the angle between the line PQ and the plane ABCD will be the angle QTq, where PQ is produced to meet the plane at T, and qT is its projection.

The angle $\mathsf{Q}q\mathsf{T}$ is a right angle, since $\mathsf{Q}q$ is perpendicular to the plane **ABCD**; hence $\cos \mathsf{QT}q = \frac{\mathsf{T}q}{\mathsf{TQ}}$; $\therefore \mathsf{T}q = \mathsf{TQ} \cos \mathsf{QT}q$. Similarly, we can show that $\mathsf{T}p = \mathsf{TP} \cos \mathsf{QT}q$; thus by subtraction $\mathsf{T}q - \mathsf{T}p = (\mathsf{TQ} - \mathsf{TP}) \cos \mathsf{QT}q$; *i.e.* $pq = \mathsf{PQ} \cos \mathsf{QT}q$. This equation proves the above rule.

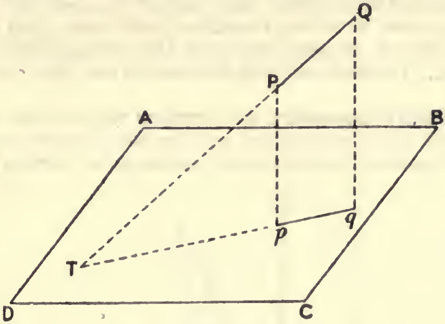


FIG. 158.

In the same way, in Fig. 157, remembering that the angle $\mathsf{RT}r = \mathsf{ST}s$ (Theorem II., Chapter XV.), we can show that $r\mathsf{T} = \mathsf{RT} \cos \mathsf{RT}r$, and that $s\mathsf{T} = \mathsf{ST} \cos \mathsf{RT}r$; whence, by adding, $rs = \mathsf{RS} \cos \mathsf{RT}r$.

Note that if a line is projected on to a plane **to which it is parallel,** the length of the projection is equal to the length of the original line.

243. On Polar Co-ordinates.—There is another method of determining the position of a point with regard to three co-ordinate planes, which is known as the method of **Polar co-ordinates.**

This is illustrated in Fig. 159. **YOZ, ZOX, XOY** are the three co-ordinate planes; **P**, the point whose position with regard to these planes is to be defined; **Q**, the projection of **P** on the plane **XOY.** Then **OQ** is the projection of **OP** on the plane **XOY.**

The polar co-ordinates of the point **P** are—

(i.) The length **OP**, which is represented by the letter r.
(ii.) „ angle **POZ**, „ „ „ „ θ.
(iii.) „ „ **QOX**, „ „ „ „ ϕ.*

The point **P** is called "the point (r, θ, ϕ)."

EXAMPLES.—CXLI.

1. Find the distance from the origin of the following points: (2, 3, 4); (4, 5, 9); (−3, 2, 0); (−3, 3, −3); (−2, −8, −9).

2. Find the inclination to the axis of z of the lines joining the origin to each of the points: (2, 3, 4); (4, 5, 9); (−3, 2, 0).

3. Find the distances between the following pairs of points: (3, 4, 5), (2, 3, 4); (7, 10, 13), (2, 1, 8); (3, −2, −3), (5, −1, 3); (−3, −4, −5), (2, −1, −6); (0, 2, 4), (3, 5, 7); (3, −3·5, −1), (4, −3, 2).

* The Greek letter *phi*, equivalent to *ph*.

4. Find the inclination to the axis of *y* of the lines joining each pair of points in Question 3.

5. Find the lengths of the projections on the plane YOZ of the lines joining each pair of points in Question 3.

244. On the Methods of Descriptive Geometry.—We have already mentioned that a solid body cannot be adequately represented by a drawing; that is to say, measurements taken from a drawing of a solid body do not properly represent the corresponding lengths on the body itself. To overcome this difficulty is the object of Descriptive Geometry.

In **Descriptive Geometry** we represent the position of a point in space *by means of its projections on two planes at right angles.* In Fig. 160, π_1 represents a **horizontal plane,** π_2 **a vertical plane**; xx

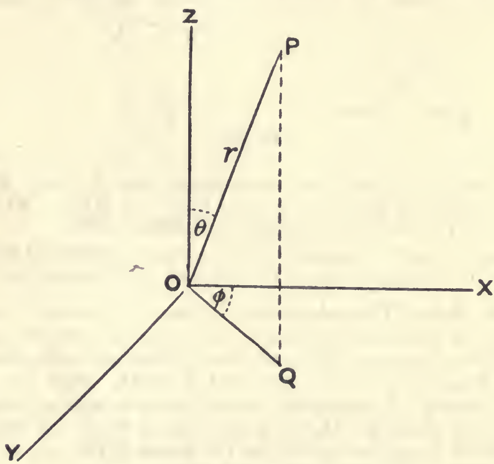


FIG. 159.

represents the intersection of these two planes, and is usually called the **axis.** If, then, we have any point A in space, its projections on π_1 and π_2 will be the points A_1, A_2 respectively; where AA_1 is perpendicular to π_1, and AA_2 is perpendicular to π_2. A_1 is called the "**horizontal projection,**" or the "**plan,**" of A; A_2 is called the "**vertical projection,**" or the "**elevation,**" of A.

If the plan and elevation of a point are known, the point is then determined; for example, in Fig. 160, any point in the line AA_1, or in AA_1 produced, would have the point A_1 for its plan; also any point in the line AA_2, or in AA_2 produced, would have the point A_2 for its elevation. But the point A is the only point which would have both A_1 for its plan and A_2 for its elevation.

If we now omit the point A, and suppose the plane π_2 to be turned back, round the axis xx, through a right angle (in the direction indicated by the arrow), it will form a continuous plane with π_1, and we shall obtain Fig. 161. Since π_1 and π_2 are now in one plane, *they can be accurately represented on the same sheet of paper.* We can therefore work with Fig. 161, provided that we remember that in actual fact the planes π_1 and π_2 are at right angles, as represented in Fig. 160.

This process of turning the plane π_2 back through a right angle is called "**rabatting**" the plane π_2 to π_1.

There is no need for the two planes of reference to be horizontal

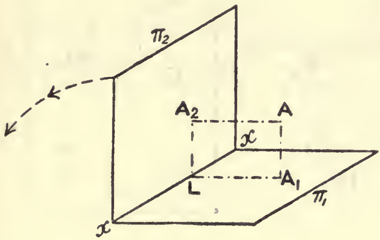


FIG. 160.

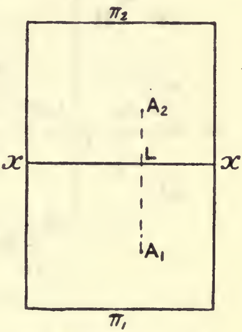


FIG. 161.

and vertical; all that is necessary is that they should be at right angles. In any case we *call* the planes the "**horizontal plane**" (H.P.) and the "**vertical plane**" (V.P.), as convenient names of reference.

Theorem.—If from the plan and from the elevation of a point we draw perpendiculars to the axis, these perpendiculars meet the axis in the same point.

EXPLANATION.—This is obvious from Fig. 160. In this figure, AA_1LA_2 is a rectangle, whose plane is perpendicular to the line xx. Thus both A_1L and A_2L are perpendicular to xx. Thus, when we have rabatted the plane π_2, as in Fig. 140, A_1L and A_2L are still perpendicular to xx.

Note that A_2L represents the height of A above π_1, for it is equal to AA_1; also that A_1L represents the distance of A in front of π_2, for it is equal to AA_2.

245. On Points in the Different Compartments.—If the planes π_1 and π_2 be produced through xx, as in Fig. 162, we divide the whole of space into four compartments. These are usually called the Ist, IInd, IIIrd, and IVth compartments, as marked in the figure. In this figure, when π_2 is rabatted (as in Fig. 163), it coincides with π_1, and both planes extend on both sides of xx.

It will be seen that, in Fig. 163, *the plan of a point may fall below or above xx, and the elevation may fall above or below xx*, according to the compartment in which the point is situated. Thus if we take the point A in Fig. 162, which lies in the compartment below π_1 and to the left of π_2 (*i.e.* in compartment III.), its plan is A_1 and its elevation is A_2. Thus when we have rabatted the plane π_1 to π_2 in the direction represented by the arrows, we obtain Fig. 163, where the plan A_1 lies above xx, and the elevation A_2 lies below xx.

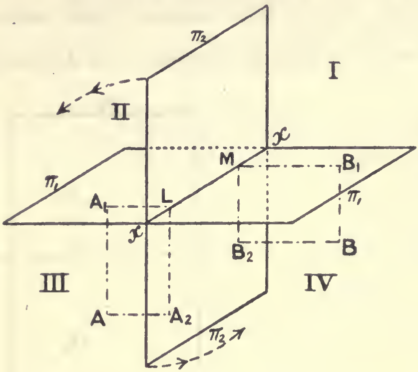


Fig. 162.

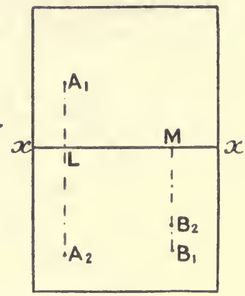


Fig. 163.

Again, consider the point B in Fig. 162, which lies in the compartment below π_1 and to the right of π_2 (*i.e.* in compartment IV.). The plan of B is B_1, and its elevation is B_2; hence, after rabatting, B_1 and B_2 both fall below xx, as in Fig. 163.

We thus obtain the following

Theorem.—If a point lies to the right of π_2 in Fig. 162, its plan falls below *xx* in Fig. 163; and conversely. If a point lies above π_1 in Fig. 162, its elevation falls above *xx* in Fig. 163; and conversely.

Note that for the future we shall represent points in space by capital letters, A, B, C, . . . ; their plans by A_1, B_1, C_1, . . . ; and their elevations by A_2, B_2, C_2, . . .

Note also that if a point lies in π_1, its elevation will lie in xx; *e.g.* the elevation of the point B_1 is M. If a point lies in π_2, its plan will lie in xx; *e.g.* the plan of B_2 is M.

The student must understand that all *accurate drawings and measurements in Descriptive Geometry are performed on the "Plan and Elevation Figures,"* such as Figs. 161 and 163; the Perspective Figures, such as Figs. 160 and 162, are merely given to explain the meaning of the "Plan and Elevation Figures."

246. On the Representation of Lines in Space.—A line can be represented in the same way as a point; that is, by its "**plan**" or "**horizontal projection;**" and its "**elevation**" or "**vertical projection.**" Thus in Fig. 164 AB is a straight line in space, A_1B_1 is its plan, and A_2B_2 is its elevation. A_1, A_2 are the plan and elevation of A ; B_1, B_2 are the plan and elevation of B.

When π_2 is rabatted to π_1, we obtain Fig. 165, in which, of course, the line AB itself is not shown, but only its plan and elevation.

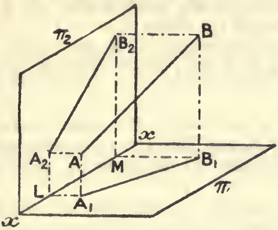


FIG. 164.

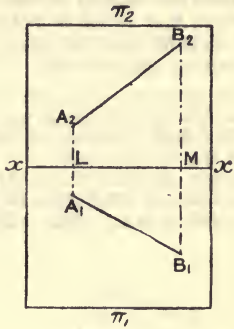


FIG. 165.

Remembering that A and B may be any two points on the line (not necessarily its extremities), we obtain the

RULE.—**The plan of a line AB is the line joining the plans A_1, B_1 of any two points A, B on the line.**

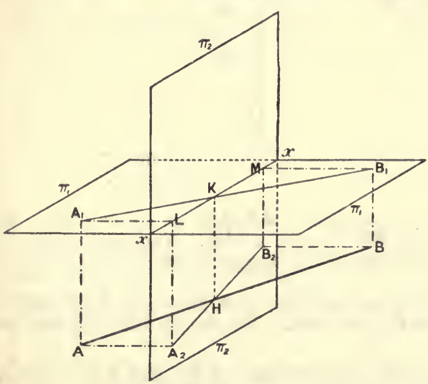


FIG. 166.

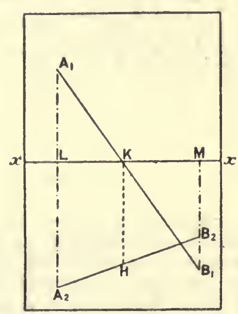


FIG. 167.

The elevation of a line AB is the line joining the elevations A_2, B_2 of any two points A, B on the line.

Fig. 166 represents a line AB joining two points in different compartments ; A_1B_1 is the plan, A_2B_2 the elevation. After rabatting, we obtain Fig. 167. Note that A_1B_1 meets xx in the point K. Note also that the point H, at which AB meets π_2, lies in the elevation A_2B_2; and that KH is perpendicular to xx.

247. To find the Length of a Line from its Projections.

Problem.—Given the projections (A_1B_1 and A_2B_2) of a line AB, find the length of the line. (See Fig. 169.)

Draw A_1P equal to A_2L, and perpendicular to A_1B_1; draw B_1Q equal to B_2M, and perpendicular to A_1B_1; join PQ. The length PQ will then be equal to AB.

EXPLANATION.—Fig. 168 represents, in perspective, the actual position of the line AB_1 and the planes π_1, π_2; then AA_1 and BB_1 are each perpendicular to the line A_1B_1, for they are each perpendicular to the plane π_1, which contains A_1B_1. (See § 166.)

If, then, the quadrilateral AA_1B_1B be rabatted to the plane π_1, we obtain the quadrilateral PA_1B_1Q, which we constructed in Fig. 169; for in Fig. 168, $PA_1 = AA_1 = A_2L$, $QB_1 = BB_1 = B_2M$.

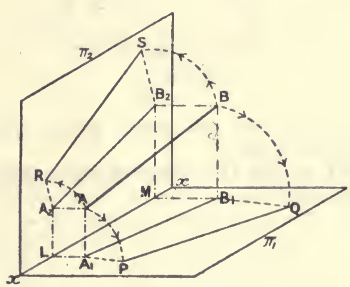


FIG. 168.

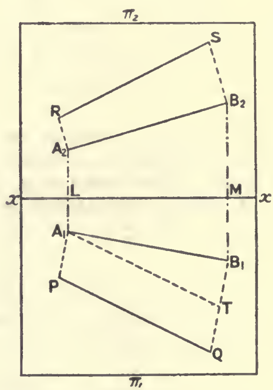


FIG. 169.

Similarly, by rabatting the quadrilateral AA_2B_2B to π_2, we obtain the quadrilateral RA_2B_2S, in which RS is equal to AB. Thus instead of the method given above, we may determine the length of AB from Fig. 169 by the following construction :—

Draw A_2R equal to A_1L, and perpendicular to A_2B_2; draw B_2S equal to B_1M, and perpendicular to A_2B_2; join RS. The length RS will then be equal to AB.

248. Problem.—Given the projections (A_1B_1 and A_2B_2) of the line AB, to determine the inclinations of AB to each of the planes of reference. (See Fig. 169.)

Make the same constructions as in § 247; then the inclination of AB to π_1 will be measured by the angle between PQ and A_1B_1; the inclination of AB to π_2 will be measured by the angle between RS and A_2B_2. To measure the angle between PQ and A_1B_1, we may either measure the angle formed by producing these lines till they meet, or we may draw A_1T parallel to PQ, and measure the angle B_1A_1T. Similarly for the angle between RS and A_2B_2.

EXPLANATION.—We have already stated, in § 242, that the inclination of a line to a plane is measured by the angle between the line and its projection on the plane. Thus in Fig. 168 the inclination of AB to π_1 is measured by the angle between AB and A_1B_1; but this is the same as the angle between PQ and A_1B_1, since the quadrilateral PA_1B_1Q is in all respects equal to the quadrilateral AA_1B_1B.

249. The more Complicated Cases of the Preceding Problems.—In the more general case, where we are using more than one of the compartments formed by the two planes, the rules are exactly the same; but it will be wiser for the student always to sketch the figure in perspective, in order to be sure of the correct application of the rules.

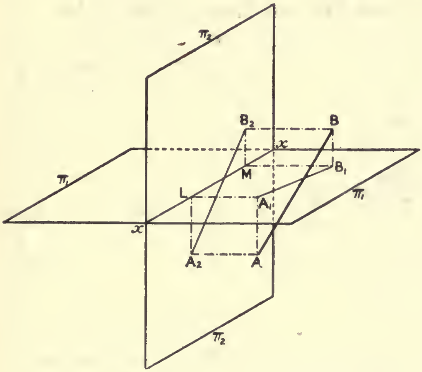


FIG. 170.

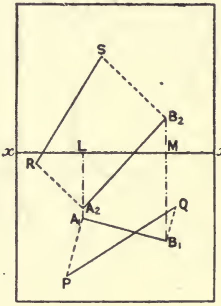


FIG. 171.

For example, if in Fig. 171 we are given the plan A_1B_1 and the elevation A_2B_2; by sketching, as in Fig. 170, the planes π_1, π_2 in their true positions, and by marking off LA_1, LA_2, MB_1, MB_2 in their true positions, we find that A is below π_1, while B is above π_1, and both A and B are to the right of π_2. Thus A and B lie on opposite sides of A_1B_1; hence, in Fig. 171, we draw A_1P and B_1Q on opposite sides of A_1B_1, in order that the figure A_1B_1QP should correspond to the figure A_1B_1BA, after the latter has been rabatted to π_1 round A_1B_1 as axis. Fig. 171 shows, without reference to Fig. 160, that A_1P and B_1Q should be on opposite sides of A_1B_1; because A_2L and B_2M (the lines whose lengths are used for A_1P and B_1Q) lie on opposite sides of *xx*.

EXAMPLES.—CXLII.

In Fig. 169, find, by geometrical construction and measurement, the length of the line AB, and its inclinations to π_1 and π_2—

1. If LM = 1 inch, LA_1 = ·3 inch, LA_2 = ·2 inch, MB_1 = ·4 inch, MB_2 = ·4 inch.
2. If LM = ·5 inch, LA_1 = ·3 inch, LA_2 = ·3 inch, MB_1 = ·2 inch, MB_2 = ·1 inch.
3. If LM = 1·2 inches, LA_1 = 1 inch, LA_2 = ·5 inch; MB_1 = 2 inches, MB_2 = 2·2 inches.
4. If the data are as in Question 1, but LA_2 is below *xx*.
5. If the data are as in Question 2, but LA_1 and MB_1 are both above *xx*.
6. If the data are as in Question 3, but LA_2 is below *xx*, and MB_1 above *xx*.

250. On Lines which pass through the Axis.—Fig. 172 represents a line AB, which meets the axis *xx* at the point A. It is obvious from this figure that the projections AB_1 and AB_2, on the planes π_1 and π_2, also pass through A.

Thus Fig. 173 shows the plan AB_1 and elevation AB_2 of AB.

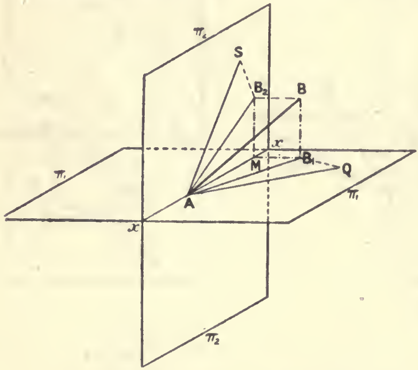


FIG. 172.

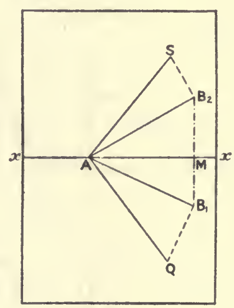


FIG. 173.

In Fig. 172, AB_1Q represents the figure AB_1B rabatted to the plane π_1. Thus in Fig. 173, if we are given the projections AB_1 and AB_2, and we wish to find the length of the line AB, we draw B_1Q equal to B_2M, and perpendicular to AB_1; then AQ will be equal to AB, since AB_1Q represents AB_1B (in Fig. 172) rabatted to π_1.

In the same way, AB_2S in Fig. 172 represents the figure AB_2B rabatted to the plane π_2.

Thus in Fig. 173 we may obtain the true length AB, by drawing B_2S equal to B_1M and perpendicular to AB_2. Then joining AS we obtain the required length.

It is obvious that these constructions are essentially the same as those in § 248, modified in accordance with the fact that, since the point **A** lies on the axis xx, it coincides with both its own plan and elevation.

In this figure the line **AB** is represented in the first compartment, but it requires but very little modification of the figure when the line **AB** lies in one of the other compartments. Note that if the line **AB** be produced through **A**, it passes into the third compartment.

251. To draw the plan and elevation of a line which passes through the axis, given its length and its inclinations to the planes of reference.

In Fig. 175 we are required to draw the plan and elevation of a line **AB** in the first compartment, given—

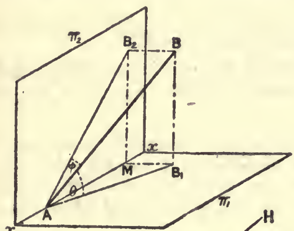
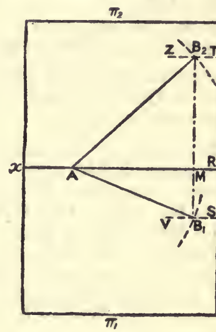


FIG. 174.



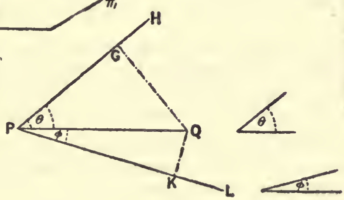


FIG. 175.

(i.) The point **A** on xx;

(ii.) That the length of the line is equal to **PQ**;

(iii.) That the inclination of the line to π_1 is equal to the angle θ, and to π_2 is equal to the angle ϕ.

We proceed as follows :—

Make the angles **QPH**, **QPL** respectively equal to θ and ϕ.

From **Q** draw **QG** perpendicular to **PH**, and **QK** perpendicular to **PL**.

In the plan and elevation figure, mark off **RS** at right angles to xx, and equal to **QK**.

Through **S** draw **SV** parallel to xx.

With centre **A**, and radius equal to **PG**, describe a circle cutting **SV** in B_1. Then B_1 will be the plan of the point **B**, and the line AB_1 will be the plan of the line **AB**.

Draw B_1M perpendicular to xx, and from B_1M produced cut off MB_2 equal to **QG**.

Then B_2 will be the elevation of **B**, and the line AB_2 will be the elevation of the line **AB**.

EXPLANATION.—In order to understand and to learn this construction, the student must sketch Fig. 174, which represents the line **AB**, the planes of reference, and the projections, in perspective.

It will be found that the triangle **QPK** in Fig. 175 gives the correct shape and size of the triangle represented in perspective by BAB_2 in Fig. 174. For **PQ** is the given length of **AB**; the angle **QPK** is equal to the given inclination of **AB** to π_2, *i.e.* to the angle BAB_2; and **QK** is drawn at right angles to **PK** (we know that BB_2 is in reality perpendicular to AB_2).

Similarly, the triangle **QPG**, in Fig. 175, gives the correct shape and size of the triangle represented in perspective by BAB_1 in Fig. 174.

Now, consider the triangle represented by AMB_1 in Fig. 174. This is the first triangle that we wish to construct in Fig. 175. And from what we have just proved, it follows that we must make $AB_1 = PG$, and $B_1M = QK$ (for $B_1M = BB_2$).

It will be seen that the construction we have followed in Fig. 175 answers this purpose, as $B_1M = RS$, and **RS** was made equal to **QK**.

Again, to obtain the correct position of B_2, we must make B_2M equal to **QG**; for, in Fig. 174, B_2M is equal to BB_1, which is known to be equal to **QG**.

It is now obvious that in Fig. 175 we could have started by constructing the triangle B_2AM. To do this we mark off **RT** equal to **QG**; draw **TZ** parallel to xx; draw a circle with centre **A**, and radius equal to **PK**, cutting **TZ** at B_2. Then draw B_2M perpendicular to xx; and from B_2M produced cut off MB_1 equal to **QK**.

Note that there is another possible position for the required line in the first compartment, viz. one in which the line **AB** and its projections lie to the left of **A**, instead of to the right. This would be obtained by working all the construction in Fig. 175 on the left side of **A** instead of on the right.

In the same way, two lines of the given length, and inclined to π_1 and π_2 at the given angles, could be constructed in each of the other eompartments. If the line is to be in the second compartment, the triangles AMB_1 and AMB_2 must both be constructed above xx. If in the third compartment, AMB_1 must be above xx, and AMB_2 below; if in the fourth compartment, AMB_1 and AMB_2 must both be below xx. In either compartment they may be constructed both to the right or both to the left of **A**.

If we are required to represent a line *of indefinite length* through a given point on the axis, inclined at given angles to the horizontal and vertical planes, we may choose any convenient length; after having obtained the projections of the line of this chosen length, these projections should then be produced in both directions.

252. Some Special Cases of the Preceding Problem.—It will be found that *in no case* can the sum of the inclinations of the required line to the horizontal and vertical planes *exceed* 90°.

(*a*) The sum of these inclinations may, however, be equal to 90°, and it will then be found that the line, its plan, and its elevation are all perpendicular to xx. This is represented in perspective in Fig. 176. Here **AB** is the line, B_1 and B_2 are the projections of **B**; AB_1, AB_2, and **AB** are all perpendicular to xx; and the figure AB_1BB_2 is a rectangle, of which **AB** is the diagonal, and whose plane is perpendicular to xx.

From the consideration of this figure we easily deduce the following

RULE.—**To draw the projections of a line in the first compartment, through a given point A on the axis, of length equal to PQ, inclined at an angle θ to the plane π_1, and at an angle ϕ to the plane π_2, when $\theta + \phi = 90°$.**

Make the angle $PQR = \theta$; from Q draw QR perpendicular to PR. From A draw AB_1 perpendicular to xx and below it, equal in length to PR. From A draw AB_2 perpendicular to xx and above it, and equal in length to QR. AB_1 is the plan, and AB_2 the elevation, of the required line. (See Fig. 177.)

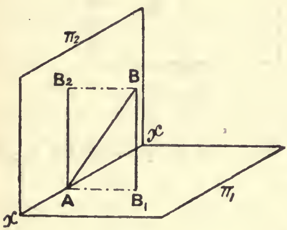
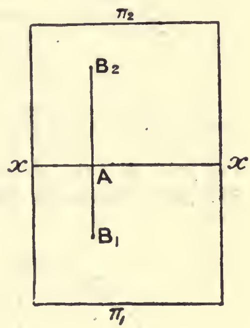


FIG. 176.



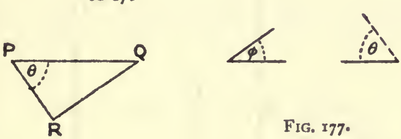


FIG. 177.

If the required line is in the second compartment, AB_1 and AB_2 must both be drawn above xx; and so on.

Only *one* such line can be drawn in each compartment.

EXPLANATION.—The triangle PQR is in all respects equal to the triangle ABB_1 in Fig. 176; for $PQ = AB$, the angle QPR = the angle BAB_1, and the angle QRP = the angle BB_1A. Thus $PR = AB_1$, and $QR = BB_1 = AB_2$.

(*b*) **The required line AB may lie in one of the planes π_1 or π_2.**

For example, Fig. 178 represents a line AB in the plane π_1 (*i.e.* inclined to π_1 at the angle 0°), and inclined to π_2 at the angle ϕ. The projection of the line on π_1 is the line AB itself; and its projection on π_2 is the line AB_2, which is a portion of the axis xx.

To draw the plan and elevation figure of this line (see Fig. 179), we need only draw AB of the given length, inclined at an angle ϕ to xx, and then draw BB_2 perpendicular to xx. AB is the required plan, and AB_2 the required elevation.

(*c*) **The required line may be** ***in one plane and perpendicular to*** *the other.* Figs. 180 and 181 represent a line in the plane π_1, and perpendicular to the plane π_2. The line AB itself is then the plan, and

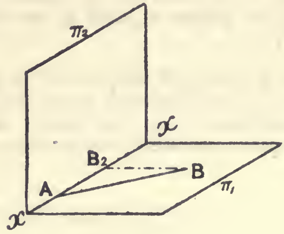


FIG. 178.

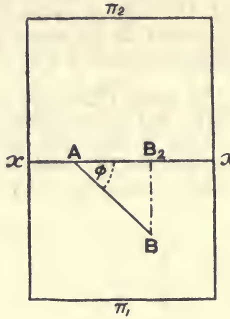


FIG. 179.

the point A is the elevation of the line. For the projection on π_2 of any point on the line AB is the point A.

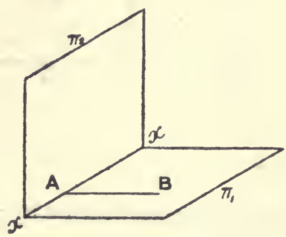


FIG. 180.

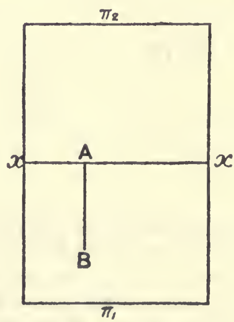


FIG. 181.

EXAMPLES.—CXLIII.

In Fig. 173, from the following data determine the length of AB and its inclination to the horizontal plane :—

1. $MAB_1 = 45°$, $AB_1 = 2·45$, $MAB_2 = 30°$, $AB_2 = 2$.
2. $MAB_1 = 45°$, $AB_1 = 1·41$, $MAB_2 = 60°$, $AB_2 = 2$.
3. $MAB_1 = 45°$, $AB_1 = 2·83$, $MAB_2 = 45°$, $AB_2 = 2·83$.
4. $MAB_1 = 60°$, $AB_1 = 2$, $MAB_2 = 60°$, $AB_2 = 2$.
5. $MAB_1 = 30°$, $AB_1 = 1·15$, $MAB_2 = 60°$, $AB_2 = 2$.

In the following examples, one extremity of the line is to lie in *xx* :—

6. Draw the projections of a line in the first compartment of length 1·25 inches, inclined at angles 25° and 36° to π_1 and π_2 respectively. Measure the lengths of the projections, and the inclinations of the projections to *xx*.

7. Draw the projections of a line in the fourth compartment of length 1·3 inches, inclined at angles 35° and 20° to π_1 and π_2 respectively. Measure the inclinations of these projections to *xx*.

8. Draw the projections of a line in the second compartment, measuring 1·45 inches in length, and inclined at angles 35° and 40° respectively, to π_1 and π_2.

9. A line in the third compartment is inclined at angles 30° and 25° to π_1 and π_2 respectively; find its projections, and measure the angle between its elevation and *xx*.

10. A line in the second compartment is inclined at angles 30° and 45° to π_1 and π_2 respectively: find its projections, and measure the angle between its plan and *xx*.

11. Draw and measure the projections of a line of length ·95 inch, inclined at angles 35° and 55° respectively to π_1 and π_2.

12. Find the lengths of the plan and elevation of a line 1·5 inches long, inclined at angles 60° and 0° to the planes π_1 and π_2 respectively.

253. To draw the projections of a line from a given point, of given length, at given inclinations to the planes of reference.

The principle which enables us to solve this problem is contained in the following

Theorem.—If two equal and parallel straight lines are projected on to the same plane, their projections are also equal and parallel.

This is shown in perspective in Fig. 182. AB and CD are equal and parallel straight lines. A_1B_1 and C_1D_1 are their projections on the plane π. Then A_1B_1 is equal and parallel to C_1D_1.

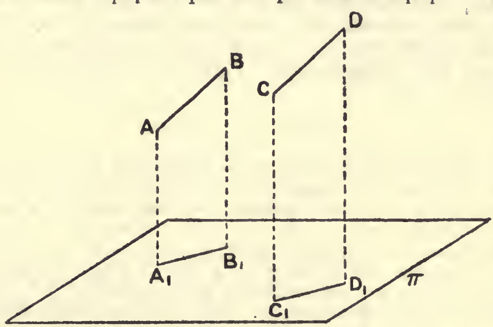


FIG. 182.

The application of this principle is shown in Fig. 183. This represents a line CD, equal and parallel to the line AB, which is drawn from a point A on the axis, such that AC_2CC_1 is a rectangle perpendicular to *xx*. AB_1 is equal and parallel to C_1D_1; for they are the projections on the same plane π_1 of the equal and parallel lines AB and CD. In the same way, AB_2 and C_2D_2 are equal and parallel; for they are the projections of the same two lines on the plane π_2.

Fig. 184 is the corresponding plan and elevation diagram.

From consideration of these figures we obtain the following

RULE.—**To draw the plan and elevation of a line CD, of given length equal to PQ, from a given point C, inclined at given angles θ and ϕ to the planes π_1 and π_2.**

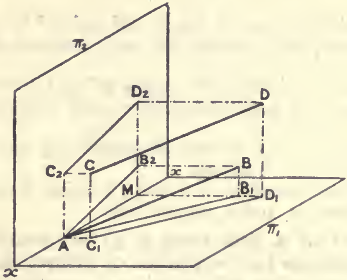


FIG. 183.

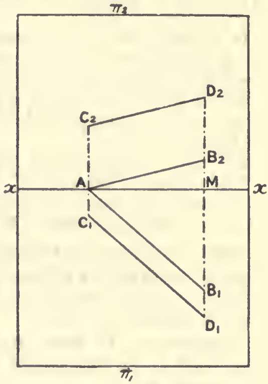


FIG. 184.

Let C_1 and C_2 be the plan and elevation of the given point C. Draw C_1AC_2 perpendicular to xx; by the rule of § 251 draw the projections AB_1 and AB_2 of a line from A inclined to the planes π_1 and π_2 at the angles θ and ϕ. Draw C_1D_1 and C_2D_2 equal and parallel to AB_1 and AB_2 respectively. C_1D_1 and C_2D_2 are the required projections.

EXAMPLES.—CXLIV.

In each of these examples measure the length AM in Fig. 175 or 184.

1. Draw the plan and elevation of a line of length 2 inches, which is drawn from a point in xx, and inclined at an angle of 20° to π_1 and 50° to π_2.

2. Draw the plan and elevation of a line of length 1·5 inches, which is drawn from a point in xx, and inclined at an angle of 30° to π_1 and π_2.

3. Draw the plan and elevation of a line of length 1·4 inches, which is drawn from a point in xx, and inclined at 40° to π_1, and at 30° to π_2.

4. Draw the plan and elevation of a line 1·2 inches long, inclined at angles of 20° and 30° respectively to π_1 and π_2, the nearer extremity of the line being ·5 inch from π_1, and ·3 inch from π_2.

5. From a point which is ·2 inch from π_1 and ·25 inch from π_2, a line is drawn of length 1·2 inches, inclined to π_1 at an angle of 25°, and to π_2 at the same angle. Draw the plan and elevation of the line.

254. On the Traces of a Line.—We have shown how a line may be represented by means of its projections; a straight line *which does not meet the axis* may also be represented by means of its "**traces**," that is, by the *points in which it meets the planes of reference.*

In Fig. 185 the line a meets π_1 at the point a_1, and π_2 at the point a_2. Thus a_1 and a_2 are the traces of the line.

RULE.—**Given the traces a_1, a_2 of a line, to find its projections.**

Draw a_1L and a_2M perpendicular to xx. Then a_1M will be the plane of the line, and a_2L its elevation. (See Figs. 185 and 186.)

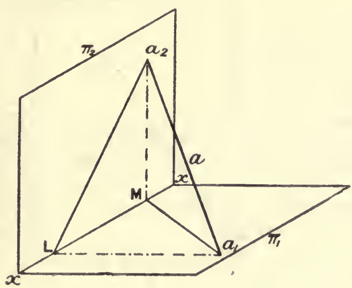


FIG. 185.

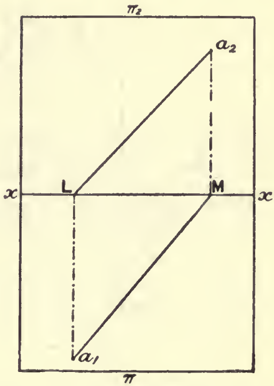


FIG. 186.

EXPLANATION.—We have seen, in § 242, that the projection of a line on a plane passes through the point where the line meets the plane. Hence the plan of the line must pass through a_1, and the elevation must pass through a_2.

Again, the projection of a line passes through the projection of any point on the line; hence the plan of this line must pass through the plan of the point a_2, that is, it must pass through M; hence a_1M is the required plan. Similarly, since the elevation of the point a_1 is L, it follows that a_2L is the required elevation.

Note that if the line passes through xx, it cannot be represented by its traces, because it meets each plane at the same point, viz. that point in which it meets xx; and this one point is not sufficient to determine the line.

255. Problem.—Given the plan A_1B_1 and the elevation A_2B_2 of a line AB, to determine the traces of the line.

Let A_2B_2, produced if necessary, cut xx at L; from L draw a line perpendicular to xx, cutting A_1B_1 in a_1; a_1 will be the horizontal trace. (See Fig. 187.)

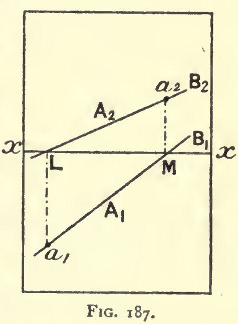


FIG. 187.

In the same way, if A_1B_1 cuts xx in M, and if the line through M perpendicular to xx cuts A_2B_2 in a_2, then a_2 will be the vertical trace.

This should need no explanation, being obviously the converse of the preceding problem.

256. On Planes and their Traces.—A plane cannot be represented by its projections on π_1 and π_2, for these projections would in general completely cover both planes; that is to say, *any point* in π_1 or π_2 would in general be the projection of some point of the given plane.

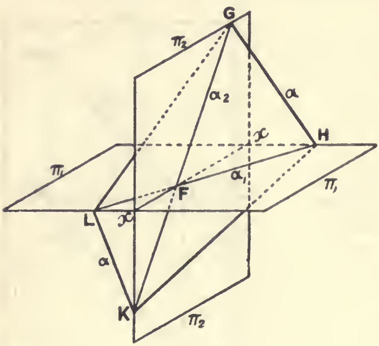


FIG. 188.

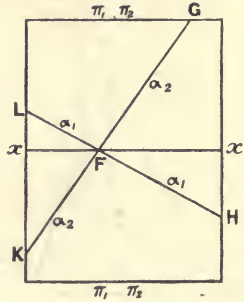


FIG. 189.

We represent a plane by its "*traces*" on π_1 and π_2, that is, by the straight lines in which the plane meets π_1 and π_2.

Fig. 188 represents a plane α, which meets the plane π_1 in the line α_1, and the plane π_2 in the line α_2; then α_1 is the **horizontal trace,** and α_2 the **vertical trace** of the plane. By rabatting the plane π_2 to π_1, we obtain Fig. 189.

CHAPTER XXVIII.

ON VECTORS.

257. On Quantities which can be represented by Straight Lines.—If a quantity possesses both *magnitude* and *direction*, it can be represented by a *straight line;* for the *magnitude* of the quantity may be represented by the *length* of the line, on some fixed "scale of representation," and the *direction* may be represented by the *direction* in which the line is drawn.

For example, if we wish to represent a force of 12 lbwt. acting in a north-easterly direction, and if we choose as our "scale of representation" 3 lbwt. to an inch, we must draw a line 4 inches long in the direction north-east.

There are many other kinds of quantities which possess both magnitude and direction; such as **velocity** (*e.g.* a velocity of 30 miles an hour due north) and **displacement,** *i.e.* change of position (*e.g.* a displacement of 1000 yards in a south-easterly direction), and others which occur in the science of Dynamics, such as **acceleration, momentum,** etc.

The direction *along the line* is indicated by the order in which the letters are quoted; for example, in Fig. 190 the line "AB" would

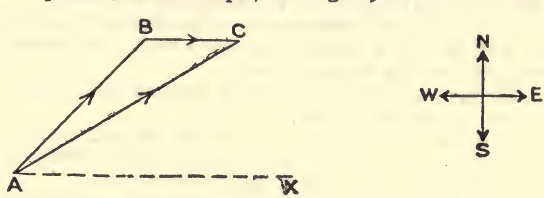


FIG. 190.

represent a quantity whose direction is north-east, the line "BA" would represent a quantity whose direction is south-west. It is convenient to indicate the direction on the diagram by an arrow-head on the line; thus in Fig. 190 an arrow-head on AB pointing toward B would indicate that the direction of the quantity is N.E.

258. On the Composition of Velocities.—The **true motion of a body is often the combined result of a number of other**

motions. For example, if a man is walking across the deck of a moving ship, his actual motion is the result of (i.) the motion of the ship, and (ii.) his motion across the deck of the ship.

Again, the ship itself may be in a current; and in this case the actual motion of the man is due to (i.) the motion of the water; (ii.) the motion of the ship through the water; (iii.) the motion of the man across the deck.

In this way a body may be said to have several simultaneous velocities; these velocities are then called the **component velocities,** and their combined effect is called the **resultant velocity.**

In Fig. 191, **S** represents the position of the ship at the beginning of a certain minute, S_1 its position at the end; and **AD** or **BC** represents the distance which the man has walked across the deck in that minute. Then **AB** represents the velocity of the ship, since it represents the distance travelled by the ship in the minute, and also gives the direction in which the ship has moved. Similarly, **AD** represents the velocity of the man across the deck. But in this minute the man has actually travelled from **A** to **C**; and therefore **AC** represents the actual velocity of the man. But since **AD** and **BC** are equal and parallel, **ABCD** is a parallelogram.

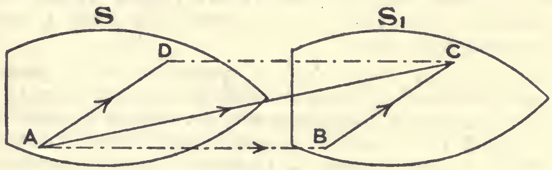


Fig. 191.

We therefore obtain the following rule for the "**Composition of Velocities,**" which is called the **Parallelogram Law**:—

Rule I.—**If the two component velocities are represented by the sides AB, AD of a parallelogram ABCD, the resultant will be represented by the diagonal AC.**

The rule may, however, be stated in another form, which in many cases is more convenient for use.

We may use the line **BC** instead of the line **AD** to represent the velocity of the man across the deck. We therefore obtain the following rule, which we may call the **Triangle Law**:—

Rule II.—**If the component velocities are represented by the sides AB, BC of a triangle, the resultant velocity is represented by the third side AC.**

Note that in applying this rule we must adhere strictly to the *directions* indicated by the order of the letters. Thus, if the component velocities were represented by **AB, CB,** the resultant velocity would *not* be represented by **AC.**

It is shown in the science of Dynamics that the resultant of two forces *which act at the same point* can also be found by either of the preceding laws.

EXAMPLE (1).—*If a bird is trying to fly north-east with a velocity of* 20 *miles an hour, and a wind is blowing from the west at the rate of* 10 *miles an hour, find the actual velocity of the bird.*

We have to find the resultant of a velocity of 20 miles an hour N.E. due to the bird's flying, with a velocity of 10 miles an hour E. due to the wind. Let us represent a velocity of 20 miles an hour by 1 inch; then a velocity of 10 miles an hour is represented by ·5 inch. We draw **AB** = 1 inch in the direction N.E., and **BC** = ·5 inch in the direction E. (see Fig. 190). Then, by the Triangle Law, **AC** will represent the resultant velocity. If we now measure the length **AC**, we shall find it to be 1·4 inches; and since 1 inch represents 20 miles an hour, 1·4 inches represent 28 miles an hour. Also, if we draw **AX** due east, by measuring the angle **XAC** we shall find it to be 30°. Thus the resultant velocity is 28 miles an hour in a direction 30° north of east.

EXAMPLE (2).—*Forces of* 3 *lbwt. and* 5 *lbwt. respectively act at a point, and their lines of action are inclined at an angle of* 60°: *find their resultant.*

Choose any convenient scale of representation, say 5 lbwt. to an inch. Then 3 lbwt. will be represented by ·6 inch. Draw **OP** of length ·6 inch, and **OQ** of length 1 inch, making the angle **POQ** = 60° (see Fig. 192). Complete the parallelogram **QOPR**, and join **OR**. Then **OP**, **OQ** represent the two forces in magnitude and direction, and therefore, by the Parallelogram Law, **OR** represents the resultant.

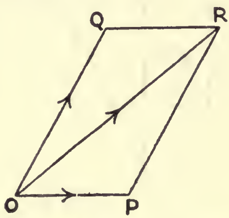


FIG. 192.

Measuring the length of **OR**, we find it to be 1·4 inches, which represents, on our present scale, a force of 7 lbwt. Measuring the angle **POR**, we find it to be 38°.

Thus the resultant is a force of 7 lbwt. acting at **O**, inclined to **OP** at an angle of 38°.

259. Extension of the Triangle Law.

RULE.—**If a series of velocities or forces are represented in magnitude and direction by the lines AB, BC, CD, DE, then the resultant velocity or force is represented in magnitude and direction by AE.**

EXPLANATION.—This is proved by repeated application of the rule in the

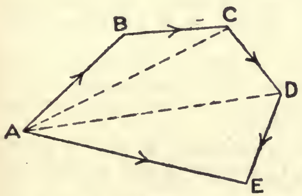


FIG. 193.

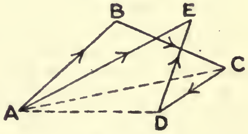


FIG. 194.

preceding paragraph. Using either Fig. 193 or Fig. 194, combining **AB** with **BC**, we obtain the resultant **AC**; combining **AC** with **CD**, we obtain the resultant **AD**; combining **AD** with **DE**, we obtain the resultant **AE**. Thus

AE represents the combined effect of the whole system. A similar theorem holds for any number of components.

EXAMPLE.—*Find the single force which would have the same effect as the following system of forces acting at the same point;* (i.) *a force of* 2 *lbwt. acting vertically up;* (ii.) *a force of* 1·5 *lbwt. acting in a direction inclined at* 30° *to the right of the upward vertical;* (iii.) *a force of* 1 *lbwt. acting horizontally to the right.*

Fig. 195 represents the three forces acting at the same point **O**; to obtain their resultant, we work as in Fig. 196. Choosing 2 lbwt. to the inch as a convenient scale of representation, we draw—

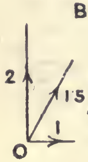


FIG. 195.

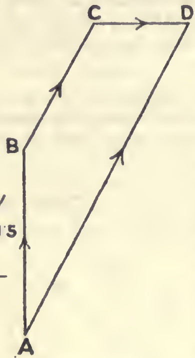


FIG. 196.

(i.) **AB** = 1 inch, vertically up; (ii.) **BC** = ·75 inch, at an angle of 30° to the right of the upward vertical; (iii.) **CD** = ·5 inch, horizontally to the right.

Then **AB**, **BC**, **CD** represent the three component forces in magnitude and direction. Thus **AD** represents the resultant force in magnitude and direction.

But **AD** will be found to measure 1·87 inches, and therefore represents a force of 2 × 1·87 = 3·74 lbwt. Also the angle **BAD** is 32°. Thus the resultant force is 3·74 lbwt. acting at an angle of 32° to the right of the upward vertical.

Note that the *order* in which the forces are represented in the diagram does not matter, the result will be the same.

Note also that, since the system of forces is equivalent to this resultant, the system could be balanced by a force equal and opposite to this resultant.

EXAMPLES.—CXLV.

1. Find the resultant velocity of a man who is walking at the rate of 4 miles an hour on the deck of a steamer which is travelling at the rate of 16 miles an hour, (i.) if the man is walking from the stern to the bow; (ii.) if the man is walking from the bow to the stern; (iii.) if the man is walking across the deck in a direction at right angles to the keel.

2. Find the resultant velocity of a bird which is trying to fly east at the rate of 35 miles an hour, while a wind is blowing from a south-westerly direction at 15 miles an hour.

3. Find the resultant of a force of 20 tonwt. acting in a direction 30° east of north, and a force of 32 tonwt. acting in a direction 10° north of east.

4. Find the resultant of two forces of magnitudes 10 lbwt. and 15 lbwt., whose lines of action are inclined at an angle of 110°.

5. Find the resultant of the following three velocities: 20 feet per second north, 25 feet per second 20° north of east, 15 feet per second 20° south of east.

6. Find the resultant of three forces of 8 lbwt., 11 lbwt., and 18 lbwt., acting at a point, if the angle between the first pair is 40°, and between the second pair 85°.

7. Find the resultant of the following four forces: 5 lbwt. north, 11 lbwt. east, 12 lbwt. 20° south of east, 7 lbwt. 25° south of west.

260. On Vectors.—The term "**vector**" is applied to velocities, forces, and all other quantities which follow the same law of combination.

Thus we may define vectors as **quantities which possess magnitude and direction, and which may be combined by the Parallelogram Law or by the Triangle Law.**

The direction of a vector, when measured by its inclination to a given line, is called the "**clinure**" or "**ort**" of the vector. The direction of the vector along this line is called its "**sense.**" Thus the clinure is represented by the slope of the line, while the sense is represented by the arrow-head.

The vector which represents the combined effect of two given vectors may be called either the "**resultant**" or the "**sum**" of the two given vectors. Thus if we have two vector quantities **P** and **Q** given in magnitude and direction, their "resultant" or "sum" will be represented by the symbol $\mathbf{P} + \mathbf{Q}$, and will be correctly determined by the application of the Parallelogram or the Triangle Law.

If we prefix a **negative sign** to a vector, it means that the vector is to be **reversed.** Thus if **P** represents a vector of magnitude 5 and direction to the right, then $-\mathbf{P}$ will represent a vector of magnitude 5 and direction to the left.

In the same way, if **P**, **Q**, and **R** are three vectors, $\mathbf{P} - \mathbf{Q}$ indicates that we are to combine **P** with **Q** reversed; $\mathbf{P} + \mathbf{Q} - \mathbf{R}$ indicates that we are to combine **P** with **Q** and the result with **R** reversed.

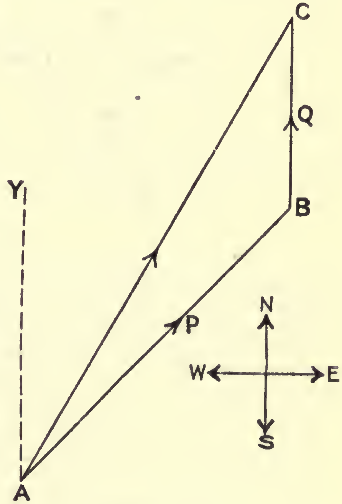


FIG. 197.

EXAMPLE.—*P is a vector of magnitude* 500, *and direction north-east; Q is a vector of magnitude* 250, *and direction north; R is a vector of magnitude* 100, *and direction west. Evaluate* (i.) $P + Q$; (ii.) $P - Q - R$.

As a convenient scale of representation, we will choose 250 to an inch; P will then be represented by a line 2 inches pointing north-east, Q by 1 inch pointing north, and R by ·4 inch pointing west.

(i.) See Fig. 197. Draw **AB** to represent P, and **BC** to represent Q; then, by the Triangle Law, **AC** represents the vector $P + Q$. Measuring **AC**, we find its length to be 2·8 inches, which represents a vector of magnitude $2{\cdot}8 \times 250 = 700$. Also, if the line **AY** points due north, we find by measuring that the angle $\mathbf{YAC} = 30°$. Thus $P + Q$ is a vector of magnitude 700, and direction 30° east of north.

(ii.) See Fig. 198. Draw **AB** to represent P; **BC** to represent $-Q$ (and therefore pointing south); and **CD** to represent $-R$ (and therefore pointing east). Then by the rule of § 259, **AD** represents the vector P − Q − R. The length of **AD** will be found to be 1·86, which represents a magnitude 1·86 × 250 = 465; and if **AX** points to the east, the angle **XAD** will be found to measure 13°. Thus P − Q − R is a vector of magnitude 465, and direction 13° north of east.

EXAMPLES.—CXLVI.

1. P is a vector of magnitude 3·5, and direction east; Q is a vector of magnitude 1·5, and direction north-east. Evaluate the vector P + Q.

2. P is a vector of magnitude 200, and direction 30° east of north; and Q is a vector of magnitude 320, and direction 10° south of west. Evaluate the vector P − Q.

3. H is a vector of magnitude 2, and direction north; K, a vector of magnitude 2·5, and direction 20° north of east; L, a vector of magnitude 1·5, and direction 20° north of west. Evaluate the vector H + K − L.

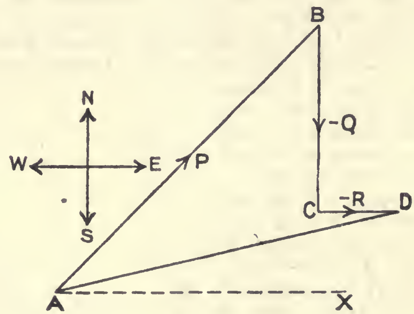


FIG. 198.

4. Evaluate the vector E − F + G − H, where E is of magnitude 500, and direction north; F of magnitude 1100, and direction west; G of magnitude 1200, and direction 20° south of east; H of magnitude 700, and direction 25° north of east.

5. In Question 1, evaluate (i.) P + 2Q; (ii.) 2P − Q.

6. In Question 2, evaluate (i.) 2P − Q; (ii.) 3P − Q.

7. In Question 3, evaluate (i.) H + 2K − L; (ii.) − 2H + 3K + 5L.

261. On Calculating the Resultant of Two Vectors.—There are three cases in which the resultant of the two vectors can be easily *calculated* without the aid of accurate drawing and measurement.

I. If the vectors are in the same direction, their resultant is their arithmetic sum.

EXPLANATION.—Two forces of magnitude 8 lbwt. and 5 lbwt. respectively, both acting vertically up, would obviously have the same effect as a single force of 13 lbwt. acting vertically up.

We can also deduce the result from the Triangle Law. For if **AB** and **BC** are both vertically up, then **ABC** is a straight line, and the resultant **AC** is equal to **AB** + **BC**. (See Fig. 199.)

II. If the vectors are in exactly opposite directions, their resultant is their arithmetic difference.

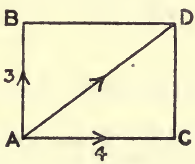
EXPLANATION.—Two forces, 8 lbwt. vertically up, and 5 lbwt. vertically down, would obviously have the same effect as a single force of 3 lbwt. vertically up.

We can also deduce the result from the Triangle Law. For if **AB** is drawn vertically up, and **BC** vertically down, then **BC** lies along **AB**, and the resultant **AC** is equal to **AB** − **BC**. (See Fig. 200.)

III. If the vectors are at right angles, we can calculate the resultant by the use of Theorem IX., Chapter XV.

EXAMPLE.—*Determine the resultant of two vectors of magnitudes* 3 *and* 4, *whose directions are at right angles.* (See Fig. 201.)

Let **AB** and **AC** represent the component vectors. Complete the parallelogram **BACD**, which in this case must be a rectangle, since the angle **A** is a right angle. Then the diagonal **AD** will represent the resultant vector.



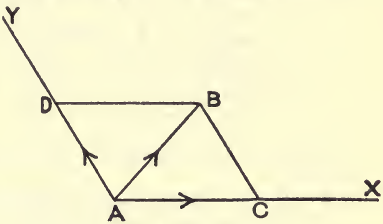


FIG. 199. FIG. 200. FIG. 201. FIG. 202.

But by Theorem IX., Chapter XV.—

$$AD^2 = AC^2 + CD^2 = AC^2 + AB^2 = 4^2 + 3^2 = 25$$

Thus **AD** = 5, which determines the magnitude of the resultant vector.

To find the direction of the resultant vector—

$$\tan DAC = DC \div AC = 3 \div 4 = \cdot 75$$

Thus from the tables the angle **DAC** = 37°.

262. On the Resolution of Vectors.—It is always possible to "**resolve**" a given vector into two components in given directions; *i.e.* to *replace the given vector by two vectors in the given directions whose combined effect would be equivalent to the original vector.*

In Fig. 202 let **AB** represent the given vector in magnitude and direction; and let **AX** and **AY** be the two given directions. Draw **BC** parallel to **AY**, and **BD** parallel to **AX**; then **AC** and **AD** will represent the required component vectors. For by the parallelogram law, the vectors represented by **AC** and **AD** will be together equivalent to the vector represented by **AB**.

If the two given directions are at right angles, the components of the vector are called its "**resolved parts**," or its "**resolutes.**" The terms "resolute" and "resolved part" are never applied to components which are *not at right angles;* though the verb "**to resolve**" may be used whether the components are at right angles or not.

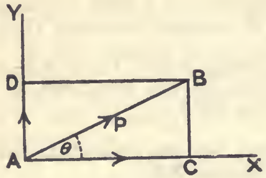


FIG. 203.

RULE.—**Given a vector P, and a line inclined to its direction at an angle θ, then the resolutes of P along and perpendicular to this line are P *cos* θ and P *sin* θ respectively.**

PROOF.—In Fig. 203 let AB represent the vector P, and let AX be the given direction, and AY perpendicular to it. Then if BC be drawn parallel to AY, and BD parallel to AX, AC and AD represent the resolutes of P along and perpendicular to AX.

But since the angle ACB is a right angle

$$\frac{AC}{AB} = \cos\theta\text{; whence } AC = AB\cos\theta = P\cos\theta$$

$$\text{also } \frac{BC}{AB} = \sin\theta\text{; whence } BC = AB\sin\theta$$

$$\therefore AD = AB\sin\theta = P\sin\theta$$

263. On the Calculator of the Resultant of any Number of Vectors.

RULE.—**To calculate the resultant of any number of vectors—**

(i.) Choose any two convenient directions at right angles, and, by resolving, replace the given system of vectors by a series of vectors along these two lines only.

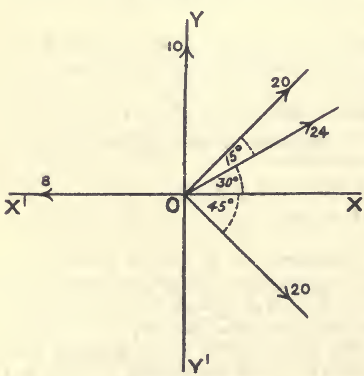


FIG. 204.

(ii.) Find the resultant of the vectors along each line; thus reducing the system to two vectors at right angles.

(iii.) Find the resultant of these two vectors at right angles.

EXAMPLE.—*Given the following vectors: P, of magnitude* 10, *and direction north; Q, of magnitude* 20, *and direction south-east; R, of magnitude* 24, *and direction* 30° *south of west; S, of magnitude* 8, *and direction west; T, of magnitude* 20, *and direction south-west; evaluate the vector* $P + Q - R + S - T$.

Remembering that the directions of R and T must be reversed, as their sign is negative, we have to combine the following vectors: 10, north; 20, south-east; 24, 30° north of east; 8, west; 20, north-east. These are represented in Fig. 204.

(i.) It will be convenient to resolve along the perpendicular lines **XOX′**, **YOY′**.

The vectors 10, north, and 8, west, are already along these lines.

The vector 20, north-east, must be replaced by its resolutes along **OX** and **OY** respectively. Since **OX** is inclined to this vector at an angle of 45°, the resolute along **OX** is 20 cos 45°, and that along **OY** is 20 sin 45° (see § 262). Similarly, the vector 24, which is inclined to **OX** at an angle 30°, gives resolutes 24 cos 30° along **OX**, and 24 sin 30° along **OY**; and the

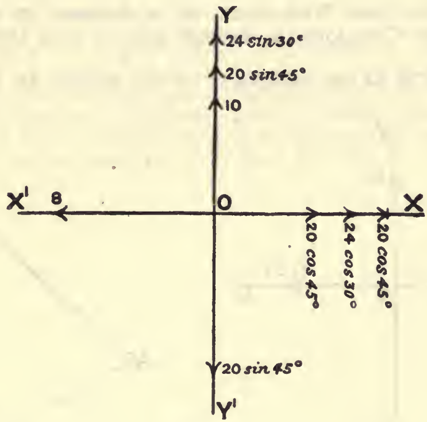


FIG. 205.

vector 20, south-east, gives resolutes 20 cos 45° along **OX**, and 20 sin 45° along **OY′**.

Thus the original system of Fig. 204 is equivalent to the system shown in Fig. 205.

(ii.) The vectors along **OX** and **OX′** are equivalent to the single vector, 20 cos 45° + 24 cos 30° + 20 cos 45° − 8 = 41·07 along **OX**. (See § 261, sections I. and II.)

Also the vectors along **OY** and **OY′** are equivalent to the single vector, 10 + 20 sin 45° + 24 sin 30° − 20 sin 45° = 22 along **OY**.

We thus reduce the system to the two vectors of Fig. 206.

(iii.) The resultant of these two vectors is found by the method of § 261, Section III.

$$\sqrt{(22^2 + 41{\cdot}07^2)} = 46{\cdot}59$$
$$(22 \div 41{\cdot}07) = {\cdot}5356 = \tan 28^\circ \text{ nearly}$$

Thus the resultant is 46·59, 28° north of east.

EXAMPLES.—CXLVII.

Determine by calculation the resultants of the following pairs of vectors :—

1. 20, north ; 30, east. **2.** 25, north ; 15, west.
3. 4, west ; 8, south. **4.** 10, east ; 4, south.

5. By resolving the vectors north or south, and east or west, work by calculation the questions in Examples CXLVI.

6. Find the resultant of five vectors whose magnitudes are 4, 5, 6, 7, and 8 respectively, if they occur in the order given, and if the angles between the successive pairs of vectors are 30°, 60,° 40,° and 100° respectively.

264. Given the Resultant of a System of Vectors, and given all the Components except one, to find this remaining Component.

RULE.—**If R is the resultant of the vectors H, K, L, and M,**

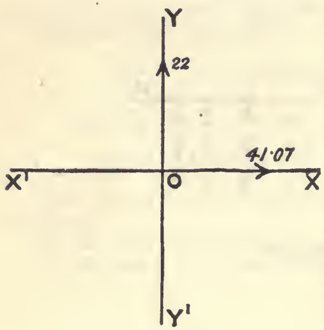


FIG. 206.

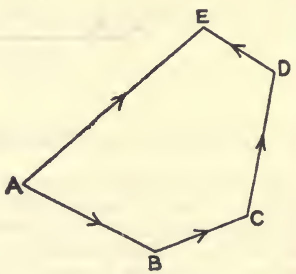


FIG. 207.

and if we are given R, K, L, and M; to find H we evaluate the vector R – K – L – M.

EXPLANATION.—Let AB, BC, CD, DE represent the component vectors H, K, L, and M ; then AE represents the resultant vector R. (See Fig. 207.)

But AB may be regarded as the resultant of a system of vectors represented by AE, ED, DC, CB, *i.e.* H may be regarded as the resultant of the vectors R, –M, –L, –K (for when the direction of a vector is reversed its sign is changed). Thus we obtain the vector equation—

$$H = R - K - L - M$$

EXAMPLE.—*The resultant of three forces is a force* 20 *lbwt. vertically up ; two of the component forces are respectively* 10 *lbwt. horizontally to the right, and* 15 *lbwt. acting at an angle of* 30° *from the upward vertical on the right of the vertical. Determine the third component.*

By the preceding rule, to determine the third component we combine the resultant with the other two components reversed. That is to say, required the resultant of 20 lbwt. vertically up, 10 lbwt. horizontally to the left, and 15 lbwt. at an angle of 30° from the downward vertical, and to the left of it.

The result of these vectors will be found, either by drawing or by calculation, to be a force of 18·85 lbwt. acting to the left of the upward vertical at an angle of about 68° from the vertical.

EXAMPLES.—CXLVIII.

1. What force must we combine with a force of 20 lbwt. acting vertically up, to obtain a resultant of 30 lbwt. acting horizontally to the left?

2. Three forces have a resultant of 50 lbwt. acting to the north; two of these forces are respectively 20 lbwt. to the east, and 30 lbwt. to the north-west: find the third force.

3. What vector must we compound with the vector 450 to the south, to obtain as resultant a vector 400 to the west?

4. A steamer is travelling to the east at the rate of 10 miles an hour; a man walking across the deck has a velocity relative to the sea of 12 miles an hour 20° south of east: find the rate and direction of his walking.

5. A bird is flying with a velocity relative to the air of 10 miles an hour to the north; the bird is actually travelling at the rate of 20 miles an hour to the north-east: find the speed and direction of the wind.

ANSWERS TO THE EXAMPLES

Examples.—I.

1. $\frac{1}{2}$; $\frac{1}{4}$; $\frac{3}{4}$; $\frac{1}{8}$; $\frac{3}{8}$; $\frac{5}{8}$; $\frac{7}{8}$. **2.** $\frac{14}{125}$; $\frac{33}{125}$; $\frac{5}{16}$; $\frac{52087}{10000}$; $\frac{4251}{250}$.
3. $\frac{2533}{125}$; $\frac{4325}{32}$; $\frac{4}{625}$; $\frac{312502}{78125}$.

Examples.—II.

1. 249·75059. **2.** 4264·671562. **3.** 77·778.
4. 879·012387. **5.** 3363·6357. **6.** 999765·872.

Examples.—III.

1. 34823·6. **2.** ·248316. **3.** ·021724. **4.** 368200.
5. ·0712201. **6.** 2·003. **7.** 121·7004. **8.** ·17309112.
9. ·0000004473. **10.** 1971·216. **11.** 5·7375. **12.** ·001518435.

Examples.—IV.

1. ·043525 ; 46·45 ; ·000384 ; 649·8 ; ·0625.
2. 8·237 ; ·250125 ; ·0667 ; ·000003 ; ·0435 ; ·044.
3. 14·87 ; ·000184 ; ·0700625 ; ·0216 ; ·4375 ; ·000008125.
4. 9·664348 ; ·0002698413 ; ·005509804 ; 84·57143 ; 12·42105 ; ·0000009411765.

Examples.—V.

1. ·043525 ; 46·45 ; ·000384 ; 64980 ; 625.
2. 148700 ; 184 ; ·700625 ; 21·6 ; 43750 ; ·8125.
3. 966·4348 ; ·002698413 ; ·005509804 ; 84·57143 ; 1·24·105 ; ·0009411765.

Examples.—VI.

1. $\frac{5}{9}$. **2.** $\frac{8}{11}$. **3.** $2\frac{9}{11}$. **4.** $5\frac{4}{27}$. **5.** $12\frac{1}{37}$. **6.** $\frac{65}{111}$.
7. $1\frac{15}{37}$. **8.** $21\frac{1}{909}$. **9.** $\frac{25}{27}$. **10.** $\frac{1}{7}$. **11.** $\frac{6}{7}$.
12. $\frac{5}{7}$. **13.** $\frac{5}{13}$. **14.** $\frac{2}{13}$. **15.** $\frac{3}{13}$. **16.** $\frac{10}{13}$.

Examples.—VII.

1. $\frac{1}{6}$. **2.** $\frac{1129}{3300}$. **3.** $21\frac{131}{330}$. **4.** $\frac{56}{185}$. **5.** $\frac{37}{44}$.
6. $3\frac{21}{275}$. **7.** $\frac{5}{148}$. **8.** $3\frac{85}{148}$. **9.** $\frac{997}{1212}$. **10.** $\frac{177}{808}$.

Examples.—VIII.

1. ·375. **2.** ·3125. **3.** ·21875. **4.** ·140625. **5.** ·4.
6. ·16. **7.** ·056. **8.** ·0192. **9.** ·6875. **10.** ·52.
11. ·0325. **12.** ·068. **13.** ·021875. **14.** $\cdot\dot{6}$. **15.** $\cdot\dot{5}$.
16. $\cdot\dot{7}$. **17.** $1\cdot\dot{6}$. **18.** $\cdot 8\dot{3}$. **19.** $\cdot 58\dot{3}$. **20.** $\cdot 41\dot{6}$.

Examples.—IX.

1. $\cdot 3\dot{8}$. **2.** $\cdot 3\dot{6}\dot{1}$. **3.** $\cdot\dot{6}\dot{3}$. **4.** $\cdot 2\dot{1}\dot{8}$. **5.** $\cdot 02\dot{1}\dot{8}$.
6. $\cdot 4\dot{8}\dot{6}$. **7.** $\cdot\dot{1}9\dot{8}$. **8.** $\cdot 08\dot{7}8\dot{3}$. **9.** $\cdot\dot{0}82\dot{5}$. **10.** $\cdot 0\dot{6}10\dot{5}$.
11. $\cdot 6\dot{7}71\dot{1}$. **12.** $\cdot 1\dot{3}782\dot{1}$. **13.** $\cdot 30\dot{9}40\dot{5}$. **14.** $\cdot\dot{4}2857\dot{1}$. **15.** $\cdot\dot{7}1428\dot{5}$.
16. $\cdot\dot{1}5384\dot{6}$. **17.** $\cdot\dot{3}8461\dot{5}$. **18.** $\cdot\dot{9}2307\dot{6}$.

Examples.—X.

1. $1\cdot 3\dot{3}4906\dot{6}$. **2.** $\cdot 1\dot{9}3366\dot{0}$. **3.** $\cdot 4\dot{6}$. **4.** $\cdot\dot{1}98\dot{0}$. **5.** $\cdot 6\dot{0}5\dot{4}$.
6. $1\cdot 0\dot{7}1428\dot{5}$. **7.** ·9207138. **8.** ·07442775. **9.** 1·381395.

Examples.—XI.

1. ·59225. **2.** $\cdot 7169\dot{2}$. **3.** 32·576. **4.** 2·001. **5.** $\cdot 0\dot{2}\dot{4}$.
6. 25·35. **7.** 2·10. **8.** ·1341 oz. **9.** May 29. **10.** ·66304.
11. 9475·375. **12.** 204 lbs.; ·444 lb. per square inch.

Examples.—XII.

1. 20 : 63. **2.** 16 : 5. **3.** 125 : 112. **4.** 25 : 6. **5.** 5 : 4. **6.** 126 : 125.
7. 12 shillings. **8.** 67·8 inches. **9.** 12·02 inches. **10.** 15 : 8 ; £200.
11. (i.) 1 : 1·75 ; (ii.) 1 : 1·19 ; (iii.) 1 : ·625 ; (iv.) 1 : ·429 ; (v.) 1 : 2·21.
12. (i.) ·4535 ; (ii.) 1·452 ; (iii.) ·625 ; (iv.) ·6410.

Examples.—XIII.

1. (i.) $\frac{1}{4}$; (ii.) $\frac{7}{20}$; (iii.) $\frac{1}{8}$; (iv.) $\frac{1}{3}$; (v.) $\frac{1}{6}$; (vi.) $\frac{3}{40}$; (vii.) $\frac{1}{80}$; (viii.) $\frac{1}{300}$; (ix.) $\frac{1}{250}$; (x.) $\frac{9}{8}$; (xi.) $\frac{11}{8}$; (xii.) $\frac{3}{800}$.

2. (i.) £1800 ; (ii.) $76\frac{1}{2}$ acres ; (iii.) 70 gallons ; (iv.) £1 ; (v.) $12\frac{1}{2}$*d.* nearly ; (vi.) 3151 children. **3.** 629. **4.** 3654 bushels.

Examples.—XIV.

1. $37\frac{1}{2}$ p.c. **2.** 35 p.c. **3.** 6·667.
4. 2·5 p.c. **5.** 14·86 p.c. nearly. **6.** 63·04 p.c.

Examples.—XV.

1. £300. **2.** 7000 lbs. **3.** 26·27 . . . pints. **4.** 1500.

5. 2·805 ozs. of nitre, ·38625 oz. of sulphur, ·52125 oz. of charcoal, ·0375 oz. of water. **6.** 30 ozs. ; 3·09 ozs. of sulphur.

Examples.—XVI.

1. 28. **2.** 21. **3.** 35. **4.** 100. **5.** 75. **6.** 55.
7. 9. **8.** 16. **9.** 36. **10.** 75. **11.** 49. **12.** 9.
13. 10. **14.** 12. **15.** 12. **16.** 12. **17.** 8. **18.** 6.

Examples.—XVII.

1. 7 : 5. **2.** 16 : 15. **3.** 7·5 p.c. **4.** 25 p.c. **5.** 39·16 p.c.
6. 39·16 p.c. of spirit, 57·17 p.c. of water, 3·67 p.c. of other ingredients.
7. 1·845 ... tons. **8.** 20·89 ... p.c. **9.** 34·16 ... p.c.

Examples.—XIX.

1. 5·459. **2.** ·025158. **3.** 11960. **4.** 60200000.
5. ·0000009125. **6.** 7·1814. **7.** 798640. **8.** 992·0.
9. 57720000. **10.** 78·4. **11.** 10295000. **12.** 1420000.

Examples.—XX.

1. ·115. **2.** ·159. **3.** 32400. **4.** 1342000.
5. ·0018913. **6.** ·861393. **7.** 113310. **8.** 1787.
9. ·65845. **10.** ·015187. **11.** 3825000000. **12.** ·000000001930.

Examples.—XXI.

1. 48·65. **2.** 321300000. **3.** 47110. **4.** ·0085321.
5. 239310. **6.** 3·672. **7.** 127070000. **8.** ·61193.

Examples.—XXII.

1. 225; 1225; 3025; 13225; 18225; 87025; 3980025; 9000300025.
2. 105625; 275625; 1500625; 2640625; 901500625.
3. 50625; 1500625; 9150625; 31640625.
4. 8344; 42410; 1068000; 1838000000; 1571000.

Examples.—XXIII.

1. 3, 6, 9, 12, 11, 8, 5, ·3, ·06, ·11. **2.** 3, 4, 6, 8, 10, 5, 7.
3. ·3, ·6, ·1, ·05, ·7. **4.** 3, 5, 4, 2.
5. 3, 11, 2, 2, 6, ·002, ·007, 1·1.

Examples.—XXIV.

1. 125. **2.** 287. **3.** 3870. **4.** 36900. **5.** ·441.
6. 18·741. **7.** ·00195. **8.** ·000405. **9.** 13·543. **10.** ·04382.

Examples.—XXV.

1. 1·414214. **2.** 1·732050. **3.** 2·236068. **4.** 2·449490.
5. 2·645751. **6.** 3·316624. **7.** 1·772454. **8.** 37·60604.
9. 1·206591. **10.** 7·669650.

EXAMPLES.—XXVI.

1. 25. **2.** 17. **3.** 9. **4.** 2·870. **5.** 2·483. **6** 1·334. **7.** 1·631.

EXAMPLES.—XXVII.

1. ·55426. **2.** ·01630. **3.** ·13321. **4.** ·15469. **5.** ·15708.
6. 1 furlong 38 poles 3 yards 1 foot 3·5 inches.
7. 1 rood 21 sq. poles 2 sq. yards 1 sq. foot 86·7 sq. inches.
8. 4 cwt. 3 qrs. 14·3 ozs. **9.** 2 sq. feet 94·84 sq. inches.
10. 23 cub. feet 982·02 cub. inches.

NOTE.—*All* the Answers in Examples XXVII. are approximate only.

EXAMPLES.—XXVIII.

1. 3580 millimetres. **2.** ·02572 decalitres. **3.** ·00000549 hectogrammes.
4. 52837·2 decagrammes. **5.** ·9848. **6.** ·6214. **7.** 15·09.
8. 4·199. **9.** ·4250. **10.** ·4469.

EXAMPLES.—XXIX.

1. 60. **2.** 18. **3.** 360. **4.** 250. **5.** 5. **6.** $\frac{2}{3}$. **7.** 12. **8.** 4.
9. $\frac{4}{5}$. **10.** 60. **11.** 0. **12.** 16. **13.** 0. **14.** 6. **15.** 2. **16.** 0.

EXAMPLES.—XXX.

1. 1080. **2.** 9720. **3.** 31104. **4.** 2916. **5.** 0. **6.** 1658880.
7. 729. **8.** 60$\frac{3}{4}$. **9.** 0. **10.** 120. **11.** 0. **12.** 8.

EXAMPLES.—XXXI.

1. 47·71. **2.** 3·300. **3.** ·00005010. **4.** ·1913. **5.** 194·9.

EXAMPLES.—XXXII.

1. 204. **2.** 111. **3.** 519. **4.** 95. **5.** 67. **6.** ·05946.
7. 4·800. **8.** 97·63. **9.** ·5751. **10.** 1·808. **11.** 59·29. **12.** 42·38.

EXAMPLES.—XXXIII.

1. 3. **2.** 4. **3.** 11·25. **4.** 1336. **5.** 145.
6. 5·1. **7.** 2·0236. **8.** 1·5205. **9.** 2·050. **10.** 4·624.
11. 1·838. **12.** 6·686. **13.** 2·866. **14.** ·06808. **15.** 1·784.
16. 17·92 **17.** 161·28. **18.** 1129·6768. **19.** 141·92. **20.** 661·12.

EXAMPLES.—XXXIV.

1. 8. **2.** 4. **3.** 10. **4.** 4. **5.** 351. **6.** 154. **7.** 489.
8. 1700. **9.** 13. **10.** 112. **11.** 112. **12.** 1690. **13.** 4 **14.** 132.
15. 91. **16.** 38. **17.** ·3532. **18.** 1·592. **19.** 72·21. **20.** 1·059.

EXAMPLES.—XXXV.

1. 6. **2.** -1. **3.** -19. **4.** 92. **5.** 181.
6. 1·4. **7.** 5·05. **8.** 1·3. **9.** £308.

EXAMPLES.—XXXVI.

1. $5x$. **2.** $-4ab$. **3.** $114a^2bc$. **4.** $-11x^3$. **5.** $-abcdxy$.
6. $-4x^2y^2z^2$. **7.** $7ab$. **8.** $-13b^2c^3$. **9.** $-2a^3b^3$. **10.** $20l^2m^3$.

EXAMPLES.—XXXVII.

1. $18b-2c$. **2.** $14y^2-15z^2$. **3.** $34ab-24ac+40bc-2a^2$.
4. $\frac{13}{6}a+\frac{33}{4}b-\frac{23}{4}c$. **5.** $\frac{5}{12}a^2+\frac{13}{12}b^2-\frac{1}{12}c^2$. **6.** $\frac{3}{4}abcd+\frac{8}{15}bcd-\frac{7}{4}cd-4d$.
7. $\frac{11}{6}x^3+\frac{5}{2}x^2y+\frac{8}{3}xy^2+\frac{1}{2}y^3$. **8.** $\frac{9}{10}a^2-\frac{8}{5}ab+\frac{7}{5}b^2$.
9. $a^2+5b^2-2c^2-8ab+10ac$. **10.** $14x^3+2x^2-19x-8$.
11. $\frac{13}{3}x^3+\frac{5}{2}x^2y-\frac{1}{2}xy^2-y^3$. **12.** $\frac{13}{10}a-\frac{5}{6}b+\frac{31}{28}c$.
13. $2p+q+\frac{2}{3}r$. **14.** $2a^2+2b^2+2c^2+2ab-2bc+2ca$.
15. $2a^3-\frac{5}{4}b^3+c^3+\frac{7}{2}a^2b+\frac{7}{2}ab^2$. **16.** $\frac{3}{4}a^5+\frac{4}{3}a^4+\frac{1}{2}a^3+\frac{4}{5}a^2+\frac{17}{3}$.

EXAMPLES.—XXXVIII.

1. -2. **2.** 12. **3.** -10. **4.** -3. **5.** -5. **6.** -7. **7.** $-6·4$.
8. -31. **9.** $-12·1$. **10.** 1·3. **11.** 18. **12.** 9. **13.** 7·8. **14.** $7a^2b$.
15. $-10abc$. **16.** $-7x^3$. **17.** $-4x^2y$. **18.** $-16·2xyz$.
19. $-14x^3y^3z^3$. **20.** $2·3x^2$. **21.** $2bc-3ab$. **22.** $-3ab^2-6abc$.
23. $5a^3+3a^2$. **24.** $2bc-5b^2c$. **25.** $6pq-10pqr$.

EXAMPLES.—XXXIX.

1. $2a^2+8ab-13b^2$. **2.** $2x^3-5x^2y-6xy^2+15y^3$.
3. $5p^2-22pq-3pr+24q^2$. **4.** $-2x^2-5xy-10xz+4y^2+15yz+2z^2$.
5. $-\frac{1}{4}x^3+\frac{1}{2}x^2+4x-\frac{8}{105}$. **6.** $\frac{1}{6}x^4-x^3-\frac{1}{2}x^2-\frac{13}{6}$.
7. $·12a-·6b+1·1c$. **8.** $-·4a+b-·8$.
9. $·6ab+·5bc+1·6ca$. **10.** $a^2-3ab+ac-7c^2$.
11. $-2a^2+4ac+4bc-c^2$. **12.** $-\frac{1}{6}a^2+\frac{5}{6}b^2+3bc-\frac{6}{5}c^2$.
13. $3x^2-4xy-10y^2+3z^2$. **14.** $·9x^2-xy+1·2y^2-yz-1·3z^2$.
15. $3a-2b+c-2d+3e-5f$. **16.** $-·1x^2+·9y^2+·1z^2$.
17. Either $5x^2+3xz-2y^2+7z^2$; or $-x^2-3xz+6y^2-yz+3z^2$.
18. $·198x^2+·888y^2-1·554z^2$. **19.** $-·033a^2+·23a+·871$.
20. $·1x^2-·055xy-·346xz-·1y^2-·7yz-·5z^2$.

EXAMPLES.—XL.

1. $6x^5$; $135x^5$; $2x^6$; $60x^{12}$. **2.** $6a^3b^4$; $30x^3y^6$; $\frac{1}{2}x^7y^{10}$.
3. $28abcd$; $30wxyz$; $144abcdef$. **4.** $\frac{1}{9}a^2bce$; $6a^5ce^2$; $\frac{1}{8}ab^2d^7e$.
5. $\frac{3}{5}a^3b^7c^6$; $60a^8bc^6d^2$. **6.** $\frac{9}{40}lm^5n$; $\frac{4}{7}p^2q^3r$; $\frac{3}{4}xy^8z$.
7. $\frac{1}{4}a^5bcd^3e$; $\frac{2}{25}k^2l^2m^4n$. **8.** $·2a^2b^2c^2$; $·48l^2mn^2$; $·102xy^2z$.
9. $·147a^5b^7c$; $·069xy^5z^5$. **10.** $150a^3bx^4yz^3$; $350a^3bx^7y^3z^2$.
11. $720a^8b^{12}c^{10}$. **12.** $·036a^7b^8c^5x^7y^5z^7$.

EXAMPLES.—XLI.

1. $21a^2 - 28ab + 35b^2$. **2.** $24a^3 + 48a^2b - 60ab^2 + 72b^3$.
3. $16x^5 + 12x^4 - 16x^3 - 20x^2$.
4. $200a^3b + 100ab^3 + 60abc^2 - 300a^2b^2 + 200ab^2c - 240a^2bc$.
5. $50x^7y^3 - 35x^6y^4 + 40x^5y^5 + 25x^4y^6 - 60x^3y^7$.
6. $4a^2bcde + 6ab^2cde - 8abc^2de + 10abcd^2e - 12abcde^2$.
7. $\cdot16l^4m^2 + \cdot24l^2m^4 + \cdot32l^2m^2n^2 - \cdot4l^2m^3n - \cdot48l^3m^2n - \cdot56l^3m^3$.
8. $-12p^3qx^2 - 6pq^3x^2 - 3pqr^2x^2 + 1\cdot8p^2qx^3 + 2\cdot4pq^2x^3 + 3pqrx^3$.
9. $-\cdot5x^4y^2z^2 - x^2y^4z^2 + 1\cdot5x^2y^2z^4 + x^2y^3z^3 - 1\cdot5x^3y^2z^3 + 2x^3y^3z^2$.

EXAMPLES.—XLII.

1. $a^3 - 3a^2b + 3ab^2 - b^3$. **2.** $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.
3. $a^5 - 25a^3b^2 + 95a^2b^3 - 75ab^4 - 50b^5$.
4. $32x^5 - 1$. **5.** $243x^5 + 1$. **6.** $70x^4 + 149x^3y - 270x^2y^2 - 340xy^3 + 400y^4$.
7. $64x^6 - 48x^4y^2 + 12x^2y^4 - y^6$. **8.** $\frac{1}{4}a^2 - \frac{1}{9}b^2 + \frac{2}{3}bc - c^2$.
9. $\frac{1}{9}a^2 - \frac{2}{15}a + \frac{1}{25} - \frac{1}{16}b^2 - \frac{1}{4}bc - \frac{1}{4}c^2$.
10. $\frac{1}{4}x^4 + \frac{1}{4}x^2z^2 + \frac{1}{16}z^4 - \frac{1}{9}y^4 + \frac{2}{15}xy^3 - \frac{1}{25}x^2y^2$.
11. $\cdot06x^2 + \cdot17xy + \cdot22xz + \cdot12y^2 + \cdot31yz + \cdot2z^2$.
12. $\cdot012x^5 - \cdot001x^4 - \cdot048x^3 - \cdot182x^2 - \cdot012x + \cdot135$.
13. $\cdot0003x^4 - 3$. **14.** $\cdot00243x^5 - \cdot3x$.
15. $\frac{1}{4}x^6 - \frac{1}{3}x^5 + \frac{13}{36}x^4 - \frac{4}{15}x^3 + \frac{17}{80}x^2 - \frac{19}{180}x + \frac{1}{24}$.
16. $9a^3b - 9ab^3 - 24ab^2c - 16abc^2$.
17. $12a^3bc - 60a^2b^2c + 75ab^3c - 27abc^3$.
18. $x^4 + 4x^2y^2 + 16y^4$. **19.** $x^8 + \cdot09x^4y^4 + \cdot0018y^8$.
20. $-a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2$.
21. $-\cdot0016x^4 - \cdot0081y^4 - z^4 + \cdot0072xy + \cdot18y^2z^2 + \cdot08z^2x^2$.
22. $a^3b - a^3c + b^3c - b^3a + c^3a - c^3b$.
23. $2a^3b - 3a^3c + 24b^3c - 8b^3a + 27c^3a - 54c^3b$.
24. $-2a^3b + 3a^3c + 24b^3c + 8b^3a - 27c^3a - 54c^3b$.
25. $a^4b - a^4c + b^4c - b^4a + c^4a - c^4b$.
26. $\frac{1}{2}a^3b - \frac{1}{3}a^3c + \frac{1}{24}b^3c - \frac{1}{8}b^3a + \frac{1}{27}c^3a - \frac{1}{54}c^3b$.
27. $\frac{1}{2}a^4b - \frac{1}{3}a^4c + \frac{1}{48}b^4c - \frac{1}{16}b^4a + \frac{1}{81}c^4a - \frac{1}{162}c^4b$.
28. $a^3 - b^3 - c^3 - 3abc$. **29.** $x^3 - \frac{1}{8}y^3 - \frac{1}{27}z^3 - \frac{1}{2}xyz$.
30. $\cdot027l^3 - \cdot064m^3 - \cdot125n^3 - \cdot18lmn$.

EXAMPLES.—XLIII.

1. $2a^2c$. **2.** $5b^4c^2$. **3.** $2a^3x^4$. **4.** $3x^3y^3z^3$.
5. $4ax^2y^4z^5$. **6.** $2bc^2d^3$. **7.** $-4a^2c^2$. **8.** $-4ax^5y^6$.
9. $4x^5z^2$. **10.** $-6b^2c$. **11.** $9yz^3$. **12.** $-3yz^2$.
13. $-4bd^2$. **14.** $3x^2y^2$. **15.** $-4a^2b^2$. **16.** $-\cdot2x^3y^2$.

EXAMPLES.—XLIV.

1. $3a^2y^3 - 4axy + 5x^2$.
2. $-7xz^3 + 6x^2yz^4 - 4x$.
3. $3 + 9pq - 12p^3q^3r^3$.
4. $-3r + 2pqr - 2p^2qr$.
5. $2x^2z^2 - 3x^3y^2z^5 + 4xy^2z^3$.
6. $-2a^2b^2c^2 + 3ab + 4bc^4$.
7. $-2abc + 3a^2b^2cd - abcde$.
8. $-3r^2 + 2pqr - 4p^2q^2s$.
9. $7x^2y^2 - 4xy + 10$.
10. $3x^2y^2 - \cdot 2xyz - \cdot 5z^2$.

EXAMPLES.—XLV.

1. $-\cdot 23x^6 - x^5 - \cdot 182x^4 + \cdot 07x^3 + 3x^2 - \cdot 3x + \cdot 5$.
2. $\frac{1}{2}x^5 + 5x^3 - 5x^2 - \frac{1}{3}x + 5$.
3. $dx^5 + bx^4 + ax^3 - cx^2 - ex + f$.
4. $x^3 - 3xyz + y^3 + z^3$.
5. $x^4 - 2x^2y^2 - 2x^2z^2 + y^4 - 2y^2z^2 + z^4$.
6. $x^5 - 5x^3y^2 - 4x^3z^2 + 4x^2y^3 + x^2z^3 + 3y^5 - 2y^3z^2 - 2y^2z^3 + 5z^5$.
7. $4x^3y - 5x^3z - 2xy^3 - 3xz^3 + 4y^4 - 3z^4$.
8. $-hx^5 + fx^4y - dx^2y^3 + cxy^4 + ay^5 + by^4z + ey^2z^3 - gz^5$.

EXAMPLES.—XLVI.

1. $x^2 + 5x + 6$.
2. $x - 3$.
3. $x^2 + 3x + 2$.
4. $4x^2 + 8x - 7$.
5. $-3x^2 - 4x + 7$.
6. $2x^2 + 3x - 4$.
7. Quotient, $3x^2 - 7x + 8$; remainder, $-13x + 56$.
8. Quotient, $2x^2 - 5x + 6$; remainder, $-39x + 72$.
9. Quotient, $4x^2 + 6x - 8$; remainder, $79x + 5$.
10. $x^2 + 10xy + 24y^2$.
11. $2x - 3y$.
12. $9x^2 + 9xy + 2y^2$.
13. $3x^3 + 2x^2 - 4x - 1$.
14. $3x^2 + 10xy - 4y^2$.
15. $a^2 + ab + ac + b^2 - bc + c^2$.
16. $a^2 + 2ab - a + 4b^2 + 2b + 1$.
17. $ab + ac + b^2 + bc$.
18. $a + b + c$.
19. $-x^2 + y^2 - 2yz + z^2$.
20. $a^2 + ab + ac + b^2 + bc + c^2$.
21. $4b + 5c$.
22. $\cdot 3a - \cdot 5b$.
23. $1\cdot 2ab + 1\cdot 3c$.
24. $\cdot 2a + \cdot 3b - \cdot 4$.
25. $-\cdot 3a + \cdot 4b + \cdot 5c$.

EXAMPLES.—XLVII.

1. $a + b + c$.
2. $-2x - 3y$.
3. $-p + 4q + 11r$.
4. $-5a + 6b + 15c$.
5. $3a^2 + 8ab^2 - 10c^2 - 20bc + 5ca - 6ab$.
6. $11x^3 - 18x^2y - 14xy^2 - 10y^3 + 28x^2$.
7. $8x^2 - 3x - 59$.
8. $-6pr + 16qs$.
9. $4xy - 6xz - 12yz$.

EXAMPLES.—XLVIII.

1. $24x$.
2. $12x$.
3. $-x^2 - 4y^2 - 9z^2$.
4. $20xz$.
5. $4x^2 + 18x + 27$.
6. $12x^2 + 6x - 3$.
7. $-12x - 20$.
8. $a^2 + b^2 + c^2 - bc - ca - ab + 6b + 6c$.
9. $8x^2 - 5x + 4$.
10. $4x^2 - 3x + 15$.
11. $24x^3 + 216x$.
12. $x^2 - 4x + 3$.

Examples.—XLIX.

1. $5x^2+2x+15$. **2.** $-5x^2+18x-20$. **3.** $16x^3-44x^2-3x+21$.
4. $-a^3-b^3-c^3+a^2b+b^2c+c^2a+3ab^2+3bc^2+3ca^2+9abc$.
5. $c^3-2a^2b-b^2c-2ab^2-ca^2-5abc$. **6.** a^3+b^3.
7. $\cdot1p^3+\cdot1q^3+\cdot1r^3-\cdot3p^2q-\cdot3q^2r-\cdot3r^2p-\cdot3pq^2-\cdot3qr^2-\cdot3rp^2$.
8. $\cdot09a^2-\cdot09b^2+\cdot09c^2-\cdot06bc+\cdot18ca-\cdot06ab$.
9. $-\cdot16x^2+\cdot19x$. **10.** $a-3c$. **11.** $-4x^3-7x^2-2x$.
12. $-6a^2-2ab+9a$. **13** $-x^3-5x^2$. **14.** $10x^2y-35xy^2+30y^3$.
15. $abc+ac^2-b^2c+bc^2$. **16.** $2q^3+q^2r+2pq^2+r^2p$.

Examples.—L.

1. $30x^3y^3$. **2.** $28x^4y^3z^4$. **3.** $120a^2b^2c$. **4.** $84xyz$.
5. $36x^3y^2z^4$. **6.** $120x^2y^2z^2$. ***7.** $-\frac{x}{30}$. **8.** $-\frac{x}{30}$.
9. $\frac{67x-118y}{30}$. **10.** $\frac{a^2-2ab+b^2}{ab}$. **11.** $\frac{p^3r+q^3p+r^3q-p^2q^2-q^2r^2-r^2p^2}{pqr}$.
12. $\frac{4x-35-6x^2}{12x}$. **13.** $\frac{5x-44y}{12}$. **14.** 0.
15. $\frac{p^2r+q^2p+r^2q}{12}$. **16.** $\frac{125x-2x^2-72}{24}$.
17. $\frac{107x-79}{30}$. **18.** $\frac{a^2b-a^2c+b^2c-b^2a+c^2a+c^2b}{a^2b^2c^2}$.

Examples.—LI.

1. $\frac{4c^2}{9a^2}$. **2.** $\frac{2}{3y}$. **3.** $\frac{9d^2}{4c^2}$. **4** $\frac{16abc}{35}$.
5. $\frac{3abc}{35}$. **6.** $\frac{8}{5p^4q^4r^4s^4}$. **7.** $\frac{m^{11}}{10l^4n^7}$. **8.** $4y^2$.
9. $\frac{4a^2c^2-41b^2d^2}{30abcd}$. **10.** $\frac{8ac-120ab+45a}{30bc}$.
11. $\frac{126ac-80b+21a^2}{168ac}$. **12.** $\frac{24pq^3r-4q^5-45p^3qr}{9p^2qr^2}$.
13. $\frac{r^2s-pqr-6pqs+q^2r}{qrs}$. **14.** $\frac{18a^2b^2-45a^2c^2-5a^4}{3b^2c^2}$.
15. $\frac{80prs+80qr^2-72pqs+72q^2r+300pq^2-75pqrs}{120qrs}$.

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* Note that $-\frac{x}{30}$, $\frac{-x}{30}$, and $\frac{x}{-30}$ are equal quantities; for example, if $x=60$, each of these quantities gives the result -2.

EXAMPLES.—LII.

1. $9a^2 + 4b^2 - 12ab$. **2.** $c^2 + 25d^2 - 10cd$.
3. $4a^2 + 49c^2 - 28ac$. **4.** $4a^2 + 9b^2 + 16c^2 - 12ab + 16ac - 24bc$.
5. $9x^2 + 4y^2 + 25z^2 - 12xy + 30xz - 20yz$.
6. $9x^4 + 4x^2 + 9 - 12x^3 + 18x^2 - 12x = 9x^4 - 12x^3 + 22x^2 - 12x + 9$.
7. $25x^4 + 49x^2 + 9 - 70x^3 - 30x^2 + 42x = 25x^4 - 70x^3 + 19x^2 + 42x + 9$.
8. $4a^2 + 9b^2 + 16c^2 + 25d^2 - 12ab - 16ac + 20ad + 24bc - 30bd - 40cd$.
9. $9p^2 + 4q^2 + 25r^2 + 16s^2 - 12pq + 30pr - 24ps - 20qr + 16qs - 40rs$.
10. $x^6 + 9x^4 + 9x^2 + 25 - 6x^5 + 6x^4 - 10x^3 - 18x^3 + 30x^2 - 30x = x^6 - 6x^5 + 15x^4 - 28x^3 + 39x^2 - 30x + 25$.
11. $4x^6 + x^4 + 4x^2 + 25 - 4x^5 + 8x^4 - 20x^3 - 4x^3 + 10x^2 - 20x = 4x^6 - 4x^5 + 9x^4 - 24x^3 + 14x^2 - 20x + 25$.
12. $\cdot04x^2 + \cdot09y^2 + \cdot12xy$. **13.** $\cdot25x^2 + \cdot16y^2 - \cdot4xy$.
14. $\cdot04x^2 + \cdot09y^2 + \cdot16z^2 - \cdot12xy + \cdot16xz - \cdot24yz$.
15. $4p^2 + \cdot25q^2 + \cdot0625r^2 - 2pq + pr - \cdot25qr$.

EXAMPLES.—LIII.

1. $3a - b$. **2.** $4a + 3bc$. **3.** $2x - 5y$.
4. $9ab - 5c$. **5.** $5p + 7qr$. **6.** $3x^2 - 5x + 10$.
7. $5x^2 + 3xy - 7y^2$. **8.** $7x^2 + 2x - 4$. **9.** $3a - 2b + c$.
10. $3p + 5q - 7r$. **11.** $5x - 4y + 3z$. **12.** $\cdot3x - \cdot2y + \cdot4z$.
13. $2x + \cdot3y - \cdot1z$. **14.** $3x - 2$. **15.** $4x + 5$.

EXAMPLES.—LIV.

1. 0. **2.** 0. **3.** -155. **4.** -225. **5.** $-2\dot{6}4$.
6. $1295\frac{3}{4}$. **7.** $\frac{5}{6}$. **8.** $-5\frac{4}{5}$. **9.** $-1\cdot56$. **10.** $6\cdot66$.
11. $-\cdot7425$. **12.** $-\cdot17$. **13.** $-2\cdot2\dot{6}$. **14.** $28\cdot2$. **15.** $3\cdot39$.

EXAMPLES.—LV.

1. 7. **2.** 25. **3.** $1\cdot5$. **4.** $\cdot0\dot{6}$. **5.** 5. **6.** 7.
7. 5. **8.** $2\cdot5$. **9.** $\cdot\dot{3}$. **10.** 15. **11.** 28. **12.** 56.
13. $7\cdot5$. **14.** $1\cdot\dot{1}$. **15.** 10. **16.** 21. **17.** 27. **18.** 40.
19. 28. **20.** 2. **21.** 6. **22.** 4. **23.** 15. **24.** 15.

EXAMPLES.—LVI.

1. 13. **2.** 5. **3.** 3. **4.** 5. **5.** 2. **6.** $3\cdot25$. **7.** 10. **8.** 2. **9.** 6.
10. This equation is impossible; *i.e.* no value of x will satisfy the equation.
11. $-\frac{4}{13}$. **12.** 5. **13.** ± 2. **14.** 0. **15.** 3. **16.** $21\cdot5$. **17.** -2.
18. This equation is true for *any* value of x; for each side of the equation reduces to -4. An equation which is true for all values of the letter or letters involved, is called an "identity."
19. $-\cdot5$. **20.** ± 3. **21.** -3. **22.** ± 1. **23.** ± 2. **24.** $\cdot96$.
25. $2\frac{1}{6}$. **26.** $13\cdot75$. **27.** $\pm 1\cdot326$ nearly. **28.** $\pm 3\cdot674$ nearly.

Examples.—LVII.

1. 2. **2.** 1. **3.** 2. **4.** -2. **5.** 4. **6.** 5.
7. 1·875. **8.** 4·585 . . . **9.** $\cdot 1\dot{7}\dot{2}$. **10.** 5. **11.** 6. **12.** ± 3.
13. ± 4. **14.** $\pm 2\cdot 236$ . . . **15.** ± 2.
16. $\pm 1\cdot 414$ . . . **17.** ·06. **18.** ± 1.

Examples.—LVIII.

1. $7y - 8$. **2.** $2 - 6y$. **3.** $4 - 2y$. **4.** $9 - \cdot 8y$.
5. $\frac{1}{3}(p + 58)$. **6.** $\frac{1}{2}p$. **7.** $1\cdot 7\dot{3}p - \cdot 52$. **8.** $1\cdot 107p - 1\cdot 836$.
9. $\frac{1}{12}(13p + 40)$. **10.** 5 : 3. **11.** 7 : 5. **12.** 83 : 13.
13. 14 : 71. **14.** 83 : 76. **15.** 241 : 127. **16.** 1·747 : 1.
17. 1·124 : 1. **18.** ·866 : 1. **19.** ·8729 : 1. **20.** 1·203 : 1.
21. 2 : 5. **22.** 1·389 : 1.

Examples.—LIX.

1. 60. **2.** 36. **3.** 105. **4.** 24, 64.
5. 44, 56. **6.** 49, 56. **7.** 30, 35. **8.** 31, 19.
9. 58, 2. **10.** 15, 24. **11.** 45, 5. **12.** 75, 45.
13. 130, 70. **14.** 10, 15. **15.** 30, 50. **16.** $\frac{18}{11}, \frac{63}{22}$.
17. 80. **18.** 100. **19.** 5, 18. **20.** ·048125, ·171125.
21. ·5, 2, 1·25. **22.** 12, 18. **23.** 63. **24.** 18, 30.

Examples.—LX.

1. 8 gallons. **2.** 20 gallons. **3.** 500 gallons, 700 gallons.
4. £30 each. **5.** £60, £40. **6.** £2000. **7.** £600, £400.
8. 5 sixpences, 16 pennies. **9.** 12 shillings, 12 sixpences, 24 pennies.
10. 25 crowns, 10 half-crowns, 15 florins. **11.** 132 yards.
12. 80 feet. **13.** 400 sq. feet. **14.** 6 feet × 3 feet × 2 feet.
15. 9·5746 inches. **16.** £100. **17.** 165 yards from the starting-point.
18. 42 miles. **19.** 4 hours after the slower rider started.
20. 10 miles an hour. **21.** £800. **22.** 8·165, 4·899.

Examples.—LXI.

1. 2, 1. **2.** 1, 2. **3.** 3, 1. **4.** 1, 3.
5. 3, 2. **6.** 4, 3. **7.** 1, 1. **8.** 2, 0.

Examples.—LXII.

1. 2, 5. **2.** 2, 9. **3.** 2, 9. **4.** 2, 5. **5** 6, 6. **6.** ·1, ·1.
7. ·01, ·002. **8.** $\cdot 029\dot{3}, -\cdot 0016$. **9.** ·1023 nearly, ·1023 nearly.
10. 1, 2. **11.** 3, 5. **12.** 2, 2. **13.** $-\cdot 8, -2\cdot 2$. **14.** 3, 3.
15. 5, 3. **16.** 8·846, ·269.

EXAMPLES.—LXIII.

1. 1*s*.; 3*d*. **2.** £3. **3.** 4*d*. per lb.; 3*d*. per lb. **4.** 25*s*.; 12*s*.
5. Gain, 25*s*.; loss, 12*s*. **6.** Gain, 30*s*.; loss, 8*s*. **7.** 19·24 gms.
8. 50 miles. **9.** 35 and 30 miles an hour. **10.** 15*s*.
11. 2·4, 3·6. **12.** £320; £440. **13.** 240, 200. **14.** 35, 25.

EXAMPLES.—LXIV.

1. $A = 2(LB + LT + BT)$. **2.** $D = 2\sqrt{A \div 3{\cdot}1416}$.
3. $V = LBD - (L - 2T)(B - 2T)(D - 2T)$.
4. $S = C + \frac{PC}{100}$. **5** $P = 4\sqrt{A}$. **6.** $d = (px + z) \sim (p - y)w$.
7. $n = \frac{hl}{l - k}$. **8.** £25. **9.** 5 p.c.
10. $8\frac{1}{3}$ yards per second. **11.** 7 miles. **12.** $T = \frac{nqI}{p + q}$.

EXAMPLES.—LXV.

1. $7a(3a^2 - 4ab + 5b^2)$. **2.** $3b(3a^2 - 4ab + 5b^2)$.
3. $5x^3(3x^2 - 5xy - 20y^2)$. **4.** $4yz(3x^2 - 5xy - 20y^2)$.
5. $19x^3z^2(2y - 3x^2z^2)$. **6.** $13p^2q^3(4p^3q^2 + 7r^5)$.
7. $6x^3(x^3 + 5x^2 + 4x + 7)$. **8.** $9x^2(2x^3 - 3x^2y + 5xy^2 - 4y^3)$.
9. $2cd(3ab + 4be - 5ef)$. **10.** $17(3abc - 2abd + 5cde)$.

EXAMPLES.—LXVI.

1. $p^2 - q^2$. **2.** $9a^2 - 16b^2$. **3.** $25x^4 - 4y^2$.
4. $9l^2m^2 - 16h^2k^2$. **5.** $100 - x^2y^6z^2$. **6.** $1 - 9a^2b^2c^2d^4$.
7. $\cdot 04 - x^2y^2$. **8.** $4p^2 - 36q^2$. **9.** $2a^3b - 2ab^3$.
10. $12x^2y^2 - 147x^2z^2$. **11.** $(a + 5b)(a - 5b)$. **12.** $(7h + 4k)(7h - 4k)$.
13. $(3hk + lmn)(3hk - lmn)$. **14.** $(11p^2 + 9q)(11p^2 - 9q)$.
15. $(17x + 15yz)(17x - 15yz)$. **16.** $(4x^4 + 3y^3)(4x^4 - 3y^3)$.
17. $3y(2x + y)(2x - y)$. **18.** $3c(5ab + 4c)(5ab - 4c)$.
19. $7abc(2c + 3ad)(2c - 3ad)$. **20.** $25a^2b(2c + a)(2c - a)$.
21. $(x^2 + 9)(x + 3)(x - 3)$. **22.** $(x^4 + 16)(x^2 + 4)(x + 2)(x - 2)$.
23. $x(x^2 + 4)(x + 2)(x - 2)$. **24.** $x^3(x^2 + 25)(x + 5)(x - 5)$.

EXAMPLES.—LXVII.

1. $x^2 - x - 12$. **2.** $x^2 + 15x + 40$. **3.** $x^2 - 8x + 15$.
4. $x^2 + 9xy + 20y^2$. **5.** $x^2 + 2xz - 15z^2$. **6.** $x^2 - 3xyz + 2y^2z^2$.
7. $x^4 - x^2 - 20$. **8.** $x^4 - 10x^2y^2 + 21y^4$. **9.** $2x^2y - 6xy^2 - 36y^3$.
10. $3x^4 - 15x^3y - 108x^2y^2$. **11.** $(x + 1)(x + 2)$.
12 $(x + y)(x + 2y)$. **13.** $(x + 2)(x + 6)$. **14.** $(x + 2y)(x + 6y)$.
15. $(x - 2)(x - 3)$. **16.** $(x - y)(x - 2y)$. **17.** $(x - 7)(x + 2)$.
18. $(a + 8)(a - 3)$. **19.** $(p + 5q)(p - 4q)$. **20.** $(p - 4q)(p - 3q)$.

21. $(p - 10q)(p + 3q)$. **22.** $3y(x + 2)(x + 6)$. **23.** $5b(a - b)(a - 2b)$.
24. $4p(p + 8)(p - 3)$. **25.** $3x^2(y + z)(y - 6z)$.
26. $(x - 10)(x - 6)$. **27.** $(x + 9)(x + 6)$. **28.** $(p - 12q^2)(p + 2q^2)$.
29. $(h + 14k)(h - 2k)$. **30.** $(xy + 2z)(xy + 18z)$. **31.** $(a + 4b)^2$.
32. $(x - 6y)^2$. **33.** $(p - 5q)^2$.

EXAMPLES.—LXVIII.

1. $8x^2 + 26xy + 15y^2$. **2.** $12x^2 - 13x - 35$. **3.** $14x^2 + 45x - 14$.
4. $18x^2y^2 - 3xy - 1$. **5.** $49a^2 - 14abc + b^2c^2$. **6.** $30x^2 - 63x - 30$.
7. $9x^3y + 6x^2y - 24xy$. **8.** $(3x - 1)(x + 4)$. **9.** $(3a - 1)(a + 6)$.
10. $(4p - 1)(p - 5)$. **11.** $(2x - 3y)(x + 5y)$. **12.** $(5h + 3)(3h - 5)$.
13. $(2p + 3)(3p + 2)$. **14.** $(4p + 3)(3p + 4)$. **15.** $(3p + 5)(5p + 2)$.
16. $(5a - 3b)(a + 3b)$. **17.** $(8cd - 1)(7cd + 1)$. **18.** $(7pq + 8r)(3pq - 4r)$.
19. $(4x + 7y)(3x + 2y)$. **20.** $(7ab - 1)(5ab - 1)$. **21.** $(13 + yz)(3 - yz)$.
22. $3(3p + 4q)(2p - q)$. **23.** $2x(2a + b)(3a - 5b)$. **24.** $yz(5y + 2z)(3y + z)$.
25. $10x^2(7x - 10)(3x - 5)$. **26.** $(2a - 3b)^2$. **27.** $(3a + b)^2$.
28. $(3h - 5kl)^2$. **29.** $(7p - 1)^2$. **30.** $(5 - 7xyz)^2$.

EXAMPLES.—LXIX.

1. $\frac{3x}{5a}$. **2.** $\frac{2a}{3x}$. **3.** $\frac{5x}{7ay}$. **4.** $\frac{4b - 5c}{7d + e}$.
5. $\frac{p^2 + q^2}{3(p + 2q)}$. **6.** $\frac{2a(3b + c)}{7d(b - 3c)}$. **7.** $\frac{a - b}{a + b}$. **8.** $\frac{a - 3b}{a - 4b}$.
9. $\frac{2h + 3k}{3h - 2k}$. **10.** $\frac{3p - 2q}{5p + q}$. **11.** $\frac{4a - 5b}{4a + 5b}$. **12.** $\frac{2y(x + y)}{5x(x - 2y)}$.
13. $\frac{3x}{5y}$. **14.** $\frac{(x + y)(x - y)}{(2x - 3y)^2}$. **15.** $2x - 3y$. **16.** $\frac{2h + k}{h + 2k}$.
17. $\frac{10d}{3(c + d)}$. **18.** $\frac{3x - 2y}{4}$. **19.** 1. **20.** $\frac{2xy(2x - 3y)}{5(x + y)}$.

EXAMPLES.—LXX.

1. $\frac{x}{(x + 3y)(2x + 3y)}$. **2.** $\frac{13x - 10y}{6(x + 2y)(x - 2y)}$. **3.** $\frac{2x - 3}{(x + 2y)(x - 2y)}$.
4. $\frac{2}{a + 2}$. **5.** $\frac{h}{h^2 - 9k^2}$. **6.** $\frac{8xy}{(x - 2y)(x + 2y)}$.
7. 0. **8.** $\frac{1}{1 - 2x}$. **9.** $\frac{1}{1 + 2x}$.
10. $\frac{1}{1 - 4x^2}$. **11.** $\frac{ax + 2x^2 - ay + y^2}{(a - y)(a + 2x)}$. **12.** $\frac{2x + 3y + z}{(x + y)(y + z)}$.
13. $3q - 2p$. **14.** $\frac{x}{4x^2 - y^2}$. **15.** 0. **16.** 1.
17. $\frac{b}{a^2 + b^2}$. **18.** 1. **19.** $\frac{a^2 + b^2}{2ab}$.
20. $\frac{2p}{p^2 + q^2}$. **21.** $\frac{5x + 17}{2(x + 3)(x + 5)}$. **22.** $\frac{9(x + 1)}{(4x - 5)(5x - 4)^2}$.

EXAMPLES.—LXXI.

1. -20. **2.** -5. **3.** 3. **4** $\frac{5}{7}$. **5.** 9. **6.** ± 5. **7.** 21. **8.** $-\frac{4}{7}$.
9. 3. **10.** 5. **11.** $13\cdot\dot{2}$. **12.** 1. **13.** 4. **14.** $-\cdot 2$. **15.** $\cdot 2$. **16.** $\cdot\dot{6}$.

EXAMPLES.—LXXII.

1. 30. **2.** 210. **3.** $12\sqrt{35}$. **4.** $30\sqrt{30}$. **5.** $48\sqrt{3}$.
6. $168\sqrt{2}$. **7.** $72\sqrt{5}$. **8.** $72\sqrt{3}$. **9.** $24\sqrt[3]{12}$. **10.** $50\sqrt[3]{14}$.
11. 40. **12.** 120. **13.** 80. **14.** $48\sqrt[3]{3}$. **15.** $24\sqrt[3]{6}\sqrt[5]{30}$.
16. $60\sqrt[3]{4}\sqrt[5]{9}$. **17.** $360\sqrt{6}$. **18.** $120\sqrt[5]{8}$. **19.** $288\sqrt{15}$. **20.** 48.
21. ab^2c^2. **22.** $p^2q^2r^2$. **23.** $abcp\sqrt{q}$. **24.** $24abc$. **25.** $bd\sqrt[3]{ace}$.
26. pq^2. **27.** $xy\sqrt{xy}\sqrt[3]{x^2y}$. **28.** $24a\sqrt[3]{bc}$. **29.** 18.
30. $10\sqrt{70}$. **31.** $4\sqrt{7}$. **32.** $\frac{ab}{c}$. **33.** pq. **34.** $abc\sqrt[2]{b}$.

EXAMPLES.—LXXIII.

1. $2\sqrt{15}$. **2.** $2\sqrt{7}$. **3.** $10\sqrt{2}$. **4.** $7\sqrt{2}$. **5.** $10\sqrt{6}$.
6. $20\sqrt{5}$. **7.** $10\sqrt{5}$. **8.** $20\sqrt{2}$. **9.** $3\sqrt[3]{3}$. **10.** $2\sqrt[3]{6}$.
11. $3\sqrt[3]{4}$. **12.** $2\sqrt[3]{100}$. **13.** $3\sqrt[5]{3}$. **14.** $3\sqrt[5]{10}$. **15.** 3·4641.
16. 4·8990. **17.** 7·0711. **18.** 22·361. **19.** 97·980. **20.** 20·785.
21. 31·623. **22.** 48·083. **23.** 67·082. **24.** 41·569. **25.** 42·426.
26. 51·962. **27.** 189·74. **28.** 25·298. **29.** 72. **30.** 84·853.

EXAMPLES.—LXXIV.

1. $\frac{3\sqrt{2}}{8}$. **2.** $\frac{5\sqrt{3}}{6}$. **3.** $\frac{6\sqrt{10}}{25}$. **4.** $\frac{2\sqrt{15}}{15}$. **5.** $\frac{a\sqrt{bd}}{cd}$.
6. $\frac{\sqrt{abcd}}{cd}$. **7.** ·53033; 1·4434; ·75895; ·51640. **8.** 2·7217.
9. ·22588. **10.** 39·528. **11.** 3·0984. **12.** ·49690. **13.** ·22768.
14. 1·6075.

EXAMPLES.—LXXV.

1. $30a^9$; $30x^{10}$; $6x^6$. **2.** $10a^7$; $5p^7$; $16p$.
3. $60a^6b^7$; $36p^8q^7$; $60x^7y^{10}z^5$. **4.** $2a^4b^2c^4$; $10p^3q^7r^2$; $6x^2y^3z^3$.
5. $2ab^5c$; $4p^2q^8r^2$; $5a^4b^3c^5$. **6.** $81a^{12}b^{12}$; $64a^6b^6$; $32a^{10}b^{10}$; $81x^4y^8z^{12}$.
7. $9a^{14}b^{16}$; $18a^6b^7$; $1024a^{14}b^{16}$. **8.** $\frac{1}{6}a^8b^8$; $3a^9b^9$; $32a^4b^4$.
9. $\frac{9c^4d^5}{2a^2b^2}$; $\frac{27x^5y^6z^{11}}{8}$.

EXAMPLES.—LXXVII.

1. 2, 5, 2, 4, 2, $\sqrt{5}$, 2. **2.** 8, 3125, 4, 256, 8, $5\sqrt{5}$, 32.
3. -5, 4, 81, -2, 49. **4.** $\frac{1}{8}, \frac{1}{9}, \frac{1}{25}, \frac{1}{16}, \frac{1}{243}$.
5. $\frac{1}{5}, \frac{1}{4}, \frac{\sqrt{5}}{5}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{32}$. **6.** 27, 32, 27, 2.

7. $\frac{1}{4}$, $\frac{\sqrt{3}}{3}$, 1, $\frac{1}{343}$, $\frac{1}{3125}$, 1, $-\frac{1}{4}$. **8.** $\frac{3}{5}$, $\frac{27}{125}$, $\frac{1}{25}$, $\frac{125}{512}$, $\frac{59049}{16807}$.
9. 4, 8, 625, $-\frac{1000}{27}$, $\frac{25}{3}$, 0. **10.** ·5, ·125, ·25, ·0625, ·015625.
11. 2, 4, 625, ·36, $\frac{125}{64}$. **12.** 2·5, $\frac{8}{27}$, $\frac{100}{81}$, $\frac{125}{216}$, $\frac{25}{3}$.

Examples.—LXXVIII.

1. 2. **2** $\sqrt{27}$. **3.** $\sqrt[60]{a^{43}}$. **4.** $\sqrt[105]{x^{76}}$. **5.** 1. **6.** 1. **7.** 8.
8. $2\sqrt{3}$. **9.** $\frac{25}{\sqrt[3]{2}}$. **10.** $bc^{\frac{1}{2}}d^{-\frac{3}{2}}$. **11.** $a^{-\frac{2}{3}}b^{-\frac{2}{3}}$.
12. $p^{\frac{7}{12}}r^{\frac{5}{12}}s^{-\frac{1}{4}}$. **13.** $x^{\frac{4}{3}}y^{-\frac{1}{4}}z^{-\frac{1}{12}}$. **14.** $x^{\frac{13}{12}}y^{-\frac{1}{3}}z^{-\frac{1}{12}}$. **15.** $a^2b^{-\frac{1}{6}}c^{-1}$.

Examples.—LXXIX.

1. $3{\cdot}0507 \times 10^7$. **2.** $2{\cdot}4993 \times 10^9$. **3.** $7{\cdot}891 \times 10^{-7}$.
4. $3{\cdot}6821 \times 10^2$. **5.** $7{\cdot}3 \times 10^{13}$. **6.** $2{\cdot}854 \times 10^{-13}$.
7. $1{\cdot}12 \times 10^{16}$. **8.** $1{\cdot}12 \times 10^{-12}$. **9.** $4{\cdot}4 \times 10^{-4}$. **10.** 7×10^{-14}.
11. $1{\cdot}99 \times 10^{22}$. **12.** $7{\cdot}60 \times 10^5$. **13.** $1{\cdot}94 \times 10^{-11}$. **14.** $3{\cdot}77 \times 10^{20}$.

Examples.—LXXX.

1. 54. **2.** 36. **3.** 27. **4.** L=M/15 ; 7·68. **5.** 6. **6.** 25. **7.** 27.
8. L = ·125952/M ; ·341$\dot{3}$. **9** 25. **10.** 50·625.
11. L = 48/M³ ; 3. **12.** 3. **13.** 51·65.
14. 2132. **15.** 10 hrs. 25 mins. **16.** 120.
17. 3·897 ins. **18.** 78·53981 sq. ins. **19.** 222·9 days.
20. 6 cub. ft. ; 25 cub. ft. **21.** 10·583 cms. **22.** ·7906 in.; ·078125 in.

Examples.—LXXXI.

1. ·8. **2.** 1. **3.** 9. **4.** 45·8$\dot{3}$ gms. **5.** ·12$\dot{2}\dot{7}$ cm.
6. 1555·2 lbwt. **7.** ·5893 in. **8.** 21·22 ins. **9.** 16 : 15.
10. 5 : 3. **11.** 9 ins. **12.** ·45 in. **13.** 1·$\dot{7}$ ins. **14.** 4 : 15.

Examples.—LXXXI (*a*).

1. $y = 2z + 4z^2$. **2.** $A = {\cdot}5B + {\cdot}3B^{-1}$. **3.** $A = {\cdot}5 + 1{\cdot}5B^{\frac{3}{2}}$.
4. $y = 3x + 2{\cdot}8^{-x}$; 3·25. **5.** 7*s*. ; 153*s*. **6.** $P = 3Q + 8R^2$.
7. $H = 2K + 6L^{-\frac{1}{2}}$. **8.** 2. **9.** (i.) 2755 ; (ii.) 2755 ; (iii.) 3000

Examples.—LXXXII.

1. 2, 4, 5, $\frac{1}{2}$, $\frac{1}{3}$, −1, −2, −3. **2.** 2, 5, 6, −1, −4, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$.
3. 2, 4, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{7}{2}$, −1, −2, $-\frac{1}{2}$, $-\frac{3}{2}$. **4.** 3, −1, −2, $\frac{1}{2}$, $\frac{3}{2}$, $-\frac{1}{2}$, −3.
5. $\frac{1}{4}$, $-\frac{1}{2}$, −2, $-\frac{1}{2}$. **6.** 2, 3, 6, −1, −2, −4.

Examples.—LXXXIII.

1. 262144.	**2** 524288.	**3.** 1048576.	**4.** 32.	**5.** 128.
6. 64.	**7.** 1048576.	**8.** 1048576.	**9.** 262144.	**10.** 65536.
11. 262144.	**12.** 256.	**13.** 32.	**14.** 32.	**15.** 8.
16. 512.	**17.** 32768.	**18.** 65536.	**19.** $\frac{1}{524288}$.	**20.** 4096.

Examples.—LXXXIV. (See Examples.—LXXXIII.)

Examples.—LXXXV.

1. $\bar{2}$·3802112. **2.** $\bar{2}$·4771213. **3.** $\bar{7}$·1637224 ; $\bar{1}$·6922820.
4. $\bar{2}$·7141535 ; $\bar{2}$·8007604 ; $\bar{1}$·7204071. **5.** $\bar{2}$·8239087. **6.** 1·8239087.

Examples.—LXXXVI.

1. 2·8176.	**2.** 2·4518.	**3.** 1·7226.	**4.** $\bar{1}$·8525.
5. $\bar{2}$·3892.	**6.** 4·4942.	**7.** $\bar{3}$·7983.	**8.** $\bar{2}$·3762.
9. $\bar{1}$·5598.	**10.** 5·9728.	**11.** $\bar{3}$·9233.	**12.** $\bar{1}$·9034.
13. $\bar{4}$·5051.	**14.** 5·5224.	**15.** 3·8908.	**16.** $\bar{7}$·0899.
17. 5·1309.	**18.** 1·8010.	**19.** $\bar{5}$·4821.	**20.** $\bar{3}$·4314.
21. 10·5119.	**22.** $\overline{11}$·6704.	**23.** $\bar{6}$·8751.	**24.** 8·8235.

Examples.—LXXXVII.

1. 19·19.	**2.** 1384.	**3.** 1·222.	**4.** ·869.
5. 7·228.	**6.** ·01403.	**7.** 2·216.	**8.** 1119.
9. ·07958.	**10.** 129·5.	**11.** ·00006822.	**12.** ·5291.
13. ·001057.	**14.** 158·5.	**15.** ·005012.	**16.** ·00001.
17. $2·417 \times 10^{10}$.	**18.** $1·726 \times 10^{-8}$.	**19.** $8·7 \times 10^{11}$.	**20.** $1·753 \times 10^{-12}$.
21. $5·888 \times 10^{7}$.	**22.** $7·727 \times 10^{-8}$.	**23.** $1·816 \times 10^{-9}$.	**24.** $2·154 \times 10^{20}$.

Examples.—LXXXVIII.

1. 12·44.	**2.** 5·306.	**3.** 270700.	**4.** ·000002122.
5. 4·447.	**6.** 14·73.	**7.** 446200.	**8.** ·01644.
9. 16600.	**10.** ·004262.	**11.** 6345000.	**12.** ·5202.
13. 1·995.	**14.** 2·797.	**15.** 2·884.	**16.** ·5715.
17. 871·8.	**18.** ·0004347.	**19.** ·001857.	**20.** 29250.
21. 1·457.	**22.** ·02333.	**23.** 60·91.	**24.** 85·84.

Examples.—LXXXIX.

1. ·5822225.	**2.** 2·3657687.	**3.** $\bar{4}$·7644750.	**4.** 7·5168658.
5. 5·3398488.	**6.** $\bar{1}$·4971509.	**7.** $\bar{3}$·5390384.	**8.** $\bar{5}$·6532125.
9. 1·5130311.	**10.** ·4471735.	**11.** $\bar{2}$·5378191.	**12.** 4·3311235.
13. 280·31.	**14.** 30802.	**15.** ·033196.	**16.** ·00034997.
17. 5236500.	**18.** 3·4892.	**19.** 193·71.	**20.** ·014066.
21. ·00016402.	**22.** 2·2166.	**23.** 2582400.	**24.** ·000020018.

Examples.—XC.

1. 12·434. **2.** 270860. **3.** 4·4477. **4.** 44640.
5. 16591. **6.** 634380000. **7.** 2·7967. **8.** ·57152.
9. ·00043444. **10.** 29252. **11.** ·023322. **12.** 85·858.

Examples.—XCI.

1. 8·81701. **2.** 117·946. **3.** 14·4743. **4.** 51·1570. **5.** 3·03746.
6. ·677560. **7.** 4·38617. **8.** 1·89146. **9.** ·170264. **10.** 8·85132.

Examples.—XCII.

1. (i.) 7·03; (ii.) 8·03; (iii.) 7·32; (iv.) 9·10; (v.) 9·77; (vi.) 8·90.
2. (i.) 1·94; (ii.) 1·09; (iii.) 1·52; (iv.) 2·97; (v.) 3·18; (vi.) 1·23.
3. (i.) 63·5; (ii.) 89·5; (iii.) 39·7; (iv.) 97·1; (v.) 71; (vi.) 1·72; (vii.) 21·5; (viii.) 7·62; (ix.) 73·0; (x.) 1·14.

Examples.—XCIII.

1. (i.) 14·1; (ii.) 27·1; (iii.) 76·3; (iv.) 3·32; (v.) 14·3.
2. (i.) ·432; (ii.) ·646; (iii.) ·141; (iv.) ·808; (v.) ·271.
3. (i.) 155; (ii.) 171; (iii.) ·666; (iv.) ·0681; (v.) 1590; (vi.) 7250; (vii.) 1450; (viii.) ·0272; (ix.) ·162; (x.) ·920.

Examples.—XCIV.

1. 23·56. **2.** 452,600. **3.** 4483. **4.** 3671.
5. ·000009722. **6.** ·000000001462. **7.** 63·31. **8.** ·00005294.
9. 76760000000. **10.** ·001067. **11.** ·02081. **12.** ·02248.

Examples.—XCV.

1. ·5903. **2.** 8·572. **3.** 1·526. **4.** ·01524.
5. ·833. **6.** 1·426. **7.** 8·134. **8.** ·2253.

Examples.—XCVI.

1. 21·33. **2.** 3·699. **3.** 4·866. **4.** ·6558. **5.** 63·28

Examples.—XCVII.

1. 73960. **2.** 60520. **3.** 2·894. **4.** 89210.
5. ·9417. **6.** ·9374. **7.** ·1811. **8.** ·2076.
9. ·8584. **10.** $1{\cdot}55 \times 10^7$. **11.** $8{\cdot}666 \times 10^6$. **12.** ·5491.
13. ·006837. **14.** ·06554. **15.** $8{\cdot}962 \times 10^{10}$.

Examples.—XCVIII.

1. 14·29. **2.** 32·15. **3.** 68·23. **4.** 2·864.
5. 4·528. **6.** 7·59. **7.** ·001109. **8.** 1260.
9. 18·84. **10.** 57·73. **11.** 10690000. **12.** ·02408.
13. ·007615. **14.** 1122000000. **15.** 183. **16.** 578·8.

EXAMPLES.—XCIX.

1. 3·946. **2.** 16·77. **3.** 10·36. **4.** 382·5.
5. 171. **6.** 700. **7.** 71·45. **8.** 26·46.

EXAMPLES.—C.

1. 3946. **2.** 16770000. **3.** ·01036. **4.** ·0003825.
5. 171000. **6.** ·7. **7.** ·00007145. **8.** 26460000.

EXAMPLES.—CI.

1. 3·037. **2.** 3·302. **3.** 4·481. **4.** 6·542. **5.** 7·114. **6.** 9·361.
7. 2·154. **8.** 5·848. **9.** 7·047. **10.** 4·327. **11.** 32·56. **12.** 406·1.
13. 203·3. **14.** ·2951. **15.** ·9283. **16.** ·03733. **17.** 952·3.
18. ·9779. **19.** ·009283. **20.** ·07911. **21.** 44·84. **22.** ·9663.

EXAMPLES.—CII.

1. 1599. **2.** 1418. **3.** ·01018. **4.** 4·438. **5.** 4·78. **6.** 20120.
7. ·3483. **8.** 264·4. **9.** 152·5. **10.** ·2529. **11.** ·5873. **12.** 4·324.

EXAMPLES.—CIV.

1. BC = 2·7, ACB = 90°, ABC = 40°.
2. PRQ = 60°, PR = 3·76, QR = 1·88.
3. LNM = 52°, LN = 2·94, MN = 3·9.
4. FH = 3·95. **5.** 29°, 47°, 104°.
6. 39°, 48°, 93°. **7.** 1 or 2 inches.
8. A = 44°, B = 58°, C = 78°. **9.** A = 46°, B = 58°, C = 76°.
10. A = 22°, B = 38°, C = 120°. **11.** BC = 1·74, B = 36°, C = 84°.
12. BC = 1·62, B = 45°, C = 85°. **13.** BC = 2·38, B = 53½°, C = 53½°.
14. AC = 2·33, AB = 2·77, A = 64°. **15.** AC = 2·3, AB = 3·13, A = 79°.
16. AC = 3·34, BA = 2·52, B = 88°. **17.** AC = 8·22, AB = 2·07, C = 80°.
18. AC = ·73, C = 135°, B = 15°; or AC = 2·73, C = 45°, B = 105°.
19. AC = ·98, C = 127°, B = 23°; or AC = 2·48, C = 53°, B = 97°.
20. AC = 1·56, C = 90°, B = 60°. **21.** Impossible.
22. AB = ·73, B = 120°, C = 15°; or AB = 2·73, B = 60°, C = 75°.
23. AB = 1·75, B = 45°, C = 82°. **24.** AC = ·6, A = 38°, B = 22°.

EXAMPLES.—CV.

1. 3·86 inches. **2.** 20 miles. **3.** 60 yards. **4.** 4·04 inches.
5. 1·94. **6.** B = 65°, BC = ·933, AB = 2·21.
7. AC = 2·38, AB = 3·11, B = 50°. **8.** BC = 2·4, A = 53°, B = 37°.
9. BC = 1, A = 30°, B = 60°. **10.** AB = 2·5, B = 37°, A = 53°.
11. BC = 1·79, AB = 2·34, A = 50°. **12.** A = 35°, AC = 1·64, BC = 1·15.
13. B = 55°, AC = 1·31, BC = ·92. **14.** B = 70°, AC = 1·88, BC = ·684.

EXAMPLES.—CVI.

1. 1·73. **2.** 2·65, 3·46. **3.** 3·5, 3·13. **4.** 2·1, 120°. **5.** 2·4, 90°.
6. 2·73 or ·73. **7.** 60°. **8.** 2·31. **9.** 3·16. **10.** 1·41, 1·93.

EXAMPLES.—CVII.

The following are correct to two decimal places :—

1. 3·74 sq. inches. **2.** 1·77 inches.
3. 2·30 sq. inches. **4.** 1·61 sq. inches.
5. 4·96 inches, 3·97 inches, 3·31 inches ; 9·92 sq. inches.
6. 5·20 sq. inches. **7.** 7·36 sq. inches.
8. 4·54 sq. inches. **9.** 1·14 sq. inches.
10. 3·86 sq. inches. **11.** (i.) 60° ; (ii.) 120° ; (iii.) 60° each.
12. 135°, 108°. **13.** 9·51 sq. inches.
14. 4·76 sq. inches. **15.** 2·02, 3, 1·43 sq. inches.

EXAMPLES.—CVIII.

These results are correct to two decimal places :—

1. 144 cub. feet ; 192 sq. feet ; 13 feet. Diagonals, 12·65, 12·37, 5 feet.
2. 121·5 cub. inches ; 384 bricks.
3. (i.) 2400 ; (ii.) 640 ; (iii.) 25·38 feet.
4. 59,820 ozs. ; 373·87 gallons. **5.** 480 cub. inches ; 360 cub. inches.
6. 17·32 cub. inches. **7.** 72 cub. feet.
8. 3·18 cub. inches ; 15·59 sq. inches.
9. 43·3 cub. inches. **10.** 16 cub. feet.
11. 27·71 cub. feet. **12.** 13·86 cub. inches.

EXAMPLES.—CIX.

These results are correct to three decimal places :—

1. 31·416 inches ; 78·54 sq. inches. **2.** 15·582 inches ; 19·322 sq. inches.
3. 1·273 inches ; 5·093 sq. inches. **4.** ·080 sq. inch.
5. 1 sq. inch ; 28·648°. **6.** 1 sq. inch ; 114·592°.
7. 6 sq. inches ; 171·889°. **8.** 2·793 inches. **9.** 57·296°.
10. 57·296°. **11.** 1·536 sq. inches. **12.** 4363·319 sq. miles.
13. 143·241°. **14.** ·637°. **15.** 57·296 inches.

EXAMPLES.—CX.

These results are correct to three significant figures :—

1. 1·41 sq. inches ; 11·16 sq. inches. **2.** ·815 sq. inch.
3. 12·2 sq. inches. **4.** 1·14 sq. inches.
5. ·209 sq. inch. **6.** 2·53 inches.
7. 26·7 inches. **8.** 10·62 inches. **9.** 5·3 inches.
10. ·5 inch. **11.** 57 sq. inches. **12.** 17·69 sq. inches.

EXAMPLES.—CXI.

1. 1·94. **2.** 2·236. **3.** 4. **4.** 80.
5. (i.) 76°; (ii.) 38°; (iii.) 142°. **6.** 1·93; 80°. **7.** 3·76; 140°.
8. 2·09. **9.** 3.

EXAMPLES.—CXII.

1. 60°, 30°; 2, 1·73, 1·73. **3.** 1; 90°. **4.** 2·83; 39°. **5.** 3·55.

EXAMPLES.—CXIII.

These results are correct to four significant figures :—

1. 28·27 sq. inches; 14·14 cub. inches. **2.** 314·2 sq. cms.; 523·6 c.c.
3. 16·76 c.c.; 37·70 sq. cms. **4.** 2 : 1.
5. 3920 c.c.; 29·4 kilogs. **6.** 255·5 c.c. **7.** 179·1 sq. inches.
8. 103·7 sq. inches; 25·66 cub. inches. **9.** ·5642 inch.
10. 1·241 inches. **11.** 109·7 sq. cms. **12.** 53·93 cub. inches.
13. 125·7 sq. inches; 150·8 sq. inches; 125·7 cub. inches.
14. ·6366 inch. **15.** ·3183 inch. **16.** ·8921 inch.
17. 3·527 inches. **18.** 29·45 cub. feet.

EXAMPLES.—CXIV.

Correct to four significant figures :—

1. 40 cub. inches. **2.** 104·7 cub. inches. **3.** 353·4 c.c.
4. 2·387 cms. **5.** ·9772 cm. **6.** 75·4 sq. cms.
7. 13 : 5. **8.** 193·7 cub. inches. **9.** 138·2 sq. inches.
10. 1219 cub. inches. **11.** 9·382 cub. inches. **12.** $37\frac{1}{3}$ cub. inches.
13. 31·42 cub. inches; 449·2 sq. inches. **14.** 113·1 c.c.
15. 394·8 sq. inches; 197·4 cub. inches.
16. 1421 sq. inches; 1421 cub. inches.
17. ·3183 inch. **18.** ·375 cub. inch.

EXAMPLES.—CXV.

Correct to four significant figures :—

1. 633·3 ozs. **2.** 5 lbs. **3.** 136 ozs. **4.** 250 gms.
5. 2947 ozs. **6.** 2226 gms. **7.** 45 ozs. **8.** 11·52.
9. 8. **10.** ·2387. **11.** 1·28. **12.** ·9549.
13. 1·047. **14.** 48·51 cub. inches. **15.** 5 pints.
16. 1176 c.c. **17.** 2·274 cms. **18.** 36·17 metres.

EXAMPLES.—CXVI.

1. 4·32 cub. inches. **2.** 1·286 ozs. **3.** 12·5.
4. 5·2 c.c.; 19·23. **5.** 14·875 ozs. **6.** S : (S − 1); 120 gms.

Examples.—CXVII.

1. 2·52 sq. inches. **2.** ·98 sq. inch. **3.** 9·51 sq. inches.
4. 9 sq. inches. **5.** 11·31 sq. inches. **6.** 17·3 sq. inches.

Examples.—CXVIII.

1. 3·14 sq. inches. **2.** 5·37 sq. inches. **3.** 6·54 sq. inches.
4. 4·48 sq. inches. **5.** 1·41 sq. inches.

Examples.—CXIX.

1. Each of the larger portions, 2·62 sq. inches ; each of the smaller portions, 1·83 sq. inches.
2. 9·2 sq. inches. **3.** 2·46 sq. inches.
4. 12·57 sq. inches. **5.** 4·5 sq. inches.

Examples.—CXX.

Note that fairly close approximations to these answers are sufficient :—

1. £3100, £5000, £6150, £5950. **2.** 10, 16·7, 17·8 knots.
3. $1\frac{1}{4}$, $1\frac{7}{8}$, $2\frac{1}{4}$ seconds. **4.** 15·7, 20·5 feet.

Examples.—CXXI.

Fairly close approximations are sufficient :—

1. When $x = 3·46$, $y = 3·46$, a minimum.
2. When $x = 3·3$, $y = 2·73$, a minimum.
3. When $x = 6$, $y = 36$, a maximum.
4. When $x = 5·33$, $y = 85·33$, a maximum.
5. When $x = 3·08$, $y = 2·70$, a minimum.

Examples.—CXXII.

1. (3, 2) ; (−1, 3) ; (−3, −2) ; (3, −1).
2. 2·5, 7. **3.** 3·3, 8·5. **4.** −4, 2·5. **5.** 2·8, 7·2.
6. −9·9, 4·9. **7.** 5·4. **8.** 3·1. **9.** −6·0.

Examples.—CXXIII.

5. 2·76, 7·24. **6.** −9·87, 4·87. **7.** 5·44. **8.** 3·14. **9.** −6·05.

Examples.—CXXIV.

1. $y = 3 + 1·5x$. **2.** $y = -2 + ·4x$. **3.** $y = 3 - ·1x$.
4. $y = -1 - ·05x$. **5.** $y = ·6x$. **6.** $y = \frac{2}{7}x$.
7. $y = \frac{2}{3}x + \frac{2}{3}$. **8.** $y = 2·75 + ·25x$. **9.** $y = 6 - x$.
10. $y = \frac{7}{15} - \frac{1}{30}x$. **11.** $y = ·3 - ·025x$. **12.** $y = ·6 - \frac{1}{60}x$.

Fairly close approximations are sufficient in the remaining answers :—

13. $y = 2·5 + ·25x$. **14.** $y = 17·55 + ·364x$.
15. $y = 29·6 - 1·2x$. **16.** $y = 11·5 + ·05x$.

EXAMPLES.—CXXV.

(Fairly close approximations are sufficient.)

1. $y = 50 + 2{\cdot}5x$; (i.) 125 ; (ii.) 175 ; (iii.) 300.

2. $a = 5$, $b = {\cdot}3$; when $P = 8$, $Q = 7{\cdot}4$; when $P = 12$, $Q = 8{\cdot}6$; when $Q = 7{\cdot}7$, $P = 9$; when $Q = 6{\cdot}95$, $P = 6{\cdot}5$.

3. When $x = 1200$, $y = 55$; when $y = 50$, $x = 1256$; when $y = 60$, $x = 1144$; $a = 163$, $b = -{\cdot}09$.

4. $y = {\cdot}002x$; when $x = 300$, $y = {\cdot}6$; when $x = 650$, $y = 1{\cdot}3$; when $y = {\cdot}45$, $x = 225$.

5. $y = 20 + 1{\cdot}2x$; when the load is 70 tons, the horse-power is 104 ; when the load is 80 tons, the horse-power is 116.

6. $a = {\cdot}3$, $b = {\cdot}008$; when the weight is 50 grammes, the probable depression is ·7.

EXAMPLES.—CXXVI.

1. $y = 2 + 6x^{-1}$. **2.** $y = {\cdot}5 + {\cdot}125x^2$. **3.** $y^2 = 3 + 5x$.

4. $y^{-1} = 5 + x$. **5.** $y = 12 - {\cdot}5xy$. **6.** $\frac{y}{x} = 5 - {\cdot}5x$.

EXAMPLES.—CXXVII.

(Fairly close approximations will suffice.)

1. When $x = 3$, rate of increase is ·91 in y to 1 in x ; when $x = 8$, rate of increase is ·56 in y to 1 in x.

2. When $x = 5$, rate of increase is 1 in y to 1 in x ; when $x = 11$, rate of increase is 2·2 in y to 1 in x.

3. (i.) 1·15 lbwt. per inch ; (ii.) ·588 lbwt. per inch.

4. (i.) ·33 inch per second ; (ii.) ·15 inch per second ; (iii.) ·21 inch per second.

5. ·53 inch per thousand feet ; ·25 inch per thousand feet.

6. 8 feet per second, away from the starting-point ; 12 feet per second, toward the starting-point.

7. 90 feet per second ; 360 feet per second.

8. The graph is a straight line ; hence the retardation is uniform ; in each second the body loses a velocity of 5 feet per second.

9. Increasing at the rate of 5·2 miles per hour in each minute ; decreasing at the rate of 9 miles per hour in each minute.

EXAMPLES.—CXXVIII.

1. (*a*) Increase of ·16 in y for 1 in x ; (*b*) increase of ·13 in y for 1 in x ; (*c*) decrease of ·91 in y for increase of 1 in x.

2. (*a*) Increase of 3·8 in y for 1 in x ; (*b*) increase of ·2 in y for 1 in x ; (*c*) decrease of 3·4 in y for increase of 1 in x.

3. (*a*) Decrease of 1·2 in y for increase of 1 in x ; (*b*) decrease of ·075 in y for increase of 1 in x ; (*c*) increase of ·24 in y for 1 in x.

4. (*a*) 6·25 feet per second, away from the starting-point; (*b*) 5 feet per second, toward the starting-point.

5. (*a*) Increase of 3·3 feet per second per second; (*b*) increase of 4·25 feet per second per second.

Examples.—CXXIX.

1. $6x$; 18. **2.** $1{\cdot}5x^2$; 6. **3.** $\cdot 5x^3$; 4.
4. $-\cdot 214x^{-2}$; $-\cdot 856$. **5.** ·24; ·24. **6.** $\cdot 75x^{1\cdot 5}$; 6.
7. $-\cdot 75x^{-3\cdot 5}$; ·000343. **8.** $6x^5$; 192. **9.** $320x^4$; 20.
10. $8x^{-3}$; 64. **11.** x; 1, 2. **12.** $x^{-\frac{1}{2}}$; 1, ·577.
13. ·6; ·6. **14.** $4{\cdot}5x^{\cdot 5}$; 10·06. **15.** $6x - 12x^2$; 0.
16. $\cdot 2 - x^{-2}$; $-\cdot 05$. **17.** $-1{\cdot}5 - x^{-\frac{3}{2}}$; $-9{\cdot}5$. **18.** $1{\cdot}5x^{-\frac{1}{2}} + 2x^{-2}$; 4·74.

Examples.—CXXX.

1. 64 cub. inches. **2.** 927 cub. feet; 46·35 sq. feet.
3. 287 cub. inches. **4.** 22·4 feet; 2·24 feet per second.
5. 2300 feet; 25·6 feet per second.
6. 208 feet; 20·8 feet per second. 459 feet; 45·9 feet per second.

Examples.—CXXXI.

1. Lengths of segments, 1·33, 1·67. Ratio of segments = ratio of shorter sides = ·8 : 1.
2. Lengths of segments, ·9, ·21. Ratio of each pair of segments = ·429 : 1.
3. (i.) ·82; (ii.) 1·37; (iii.) 1·17. **4.** (i.) ·89; (ii.) ·51; (iii.) 1·98.
5. ·48, 1·2, ·67, ·83; each ratio = ·4 : 1.
FO : OC = FB : BC = ½AB : BC = ½AE : EC.
6. ·73, 1·34, ·53, ·4; 1·67 : 1. **7.** ·68, ·32, 1·02, ·48; 1·2 : 1.
8. 2·14. **9.** (i.) 1·4; (ii.) ·96; (iii.) 1·32.

Examples.—CXXXIII.

1. 1·5. **2.** 5·25; 4·69. **3.** 3·6; 60°. **4.** 60°. **5.** Each ratio = 1·4 : 1.

Examples.—CXXXIV.

1. 90 yards. **2.** 94 yards. **3.** 145 yards. **4.** 29°, 47°, 104°.
5. 81 feet. **6.** 3000 feet. **7.** 10 miles. **8.** 66 yards.

Examples.—CXXXV.

Correct to three significant figures.

1. 29,000 sq. yards. **2.** 4,330,000 sq. yards. **3.** 757 sq. inches.
4. 36,200 sq. yards. **5.** 1510 sq. yards. **6.** 26,600 sq. metres.

EXAMPLES.—CXXXVI.

1. (i.) $\frac{\pi}{6}$ radians, or ·5236 radian; (ii.) $\frac{\pi}{20}$ radians = ·157; (iii.) ·427; (iv.) ·3526.

2. (i.) 30°; (ii.) 72°; (iii.) 64·17°; (iv.) 33·63°; (v.) 2·006°; (vi.) 27°.

3. ·57. **4.** ·34, ·77, 2·75, 1·25, 1·34, 1·43.

5. (i.) 14·5°; (ii.) 8°; (iii.) 36°; (iv.) 25°; (v.) 64°; (vi.) and (vii.) are impossible; the sine cannot be greater than unity, as the perpendicular cannot be greater than the hypotenuse.

6. (i.) 72·5°; (ii.) 61°; (iii.) 44°; (iv.) 37°; (v.) 79·5°; (vi.) and (vii.) are impossible, as the base cannot be greater than the hypotenuse.

7. (i.) 17°; (ii.) 26·5°; (iii.) 37°; (iv.) 45°; (v.) 56°; (vi.) 74°.

8. (i.) ·94; (ii.) ·48; (iii.) 1·54.

EXAMPLES.—CXXXVII.

All angles are expressed in circular measure, unless otherwise stated.

1. ·5; ·5; 1; 2·7475; 1·0515; 1·7013. **4.** ·5 in 1; ·707 in 1.

5. y decreasing at the rate of ·707 for an increase of 1 in x; y decreasing at the rate of ·866 for an increase of 1 in x.

6. y increasing at the rate of ·5 for an increase of 1 in x; y decreasing at the rate of ·5 for an increase of 1 in x.

7. y decreasing at the rate of 1·73 for an increase of 1 in x.

8. y neither increasing nor decreasing; a minimum value of y.

9. (i.) 31°, 27°, 20°; (ii.) 29°, 60°, 63°; (iii.) 70°, 47°, 37°, 45°.

10. ·2 in 1.

11. y decreasing at the rate of ·289 for an increase of 1 in x.

EXAMPLES.—CXXXVIII.

1. (i.) $b = 45·8$, $A = 24°$, $B = 66°$; (ii.) $b = 6·25$, $A = 39°$, $B = 51°$; (iii.) $a = 24·7$, $A = 55°$, $B = 35°$; (iv.) $c = 36·05$, $A = 56°$, $B = 34°$; (v.) $c = 43$, $A = 36°$, $B = 54°$.

2. (i.) $B = 65°$, $b = 32·2$, $c = 35·5$; (ii.) $B = 55°$, $a = 17·5$, $c = 30·5$. (iii.) $B = 18°$, $b = 39$, $c = 126$; (iv.) $A = 32°$, $b = 160$, $c = 188·7$; (v.) $A = 40°$, $a = 8·39$, $c = 13·05$.

3. 38·4 feet. **4.** 34·9 yards. **5.** 52·4 feet; 17·5 feet.

6. 1860 feet. **7.** 69·5 feet. **8.** 418 yards.

9. (i.) and (ii.) 37° N. of E. **10.** 4·663 miles an hour.

EXAMPLES.—CXXXIX.

1. (i.) 636·56; (ii.) 13289. **2.** 294·28. **3.** (i.) 455·8; (ii.) ·0004822.

4. (i.) ·7431; (ii.) ·9703. **5.** (i.) 1·092; (ii.) ·3748.

6. Decrease of ·415 in y for increase of 1 in x; increase of ·227 in y for increase of 1 in x. **7.** 38° 10′.

8. Increase of ·0237 in y for 1 in x; increase of ·0174 in y for 1 in x.

9. 32° approximately. **10.** 18·8° approximately.

EXAMPLES.—CXL.

1. ·57, 1 ; 90°, 120°. **2.** 1·73 ; 70°. **3.** (2·08, 100°) ; or (2·08, 40°).
4. (1·41, 0°). **5.** 5·2 sq. inches.
6. (1·43, 1·29) ; (·513, 1·41) ; (6·3, 4·93) ; (·366, ·341).
7. (5·6, 53°) ; (1·43, 67½°) ; (2·6, 55°) ; (3·46, 60°).
8. (−·75, 1·3) ; (1·3, −·75) ; (1·41, −1·41) ; (−·41, −1·13).
9. (2·24, 116½°) ; (2·5, −53°) ; (2, −150°) ; (3·36, 116½°).
10. A semicircle. **11.** A straight line perpendicular to OX.
12. A straight line parallel to OX.

EXAMPLES.—CXLI.

1. 5·386 ; 11·05 ; 3·605 ; 5·196 ; 12·21. **2.** 42° ; 35° ; 90°.
3. 1·732 ; 11·45 ; 6·403 ; 5·916 ; 5·196 ; 3·201.
4. 55° ; 64° ; 20° ; 80° ; 55° ; 20°.
5. 1·414 ; 10·29 ; 6·082 ; 3·162 ; 4·242 ; 3·041.

EXAMPLES.—CXLII.

1. 1·02 inches, 11°, 6°. **2.** ·55, 21°, 11°. **3.** 2·31, 47°, 26°.
4. 1·17, 31°, 5°. **5.** ·55, 21°, 11°. **6.** 4·21, 40°, 45°.

EXAMPLES.—CXLIII.

1. 2·65, 22°. **2.** 2·24, 51°. **3.** 3·46, 35°. **4.** 2·65, 41°.
5. 2·08, 56°. **6.** 1·13, 1·01 ; 40°, 32°. **7.** 25°, 38°.
8. 51°, 48°. **9.** 33°. **10.** 55°. **11.** ·78, ·60. **12.** ·75, 1·5.

EXAMPLES.—CXLIV.

1. 1·09. **2.** 1·06. **3.** ·81. **4.** ·95. **5.** ·96.

EXAMPLES.—CXLV.

1. (i.) 20 miles an hour ; (ii.) 12 miles an hour ; (iii.) 16·45 miles an hour.
2. 46·8 miles an hour ; 13° N. of E. **3.** 47·4 tonwt. ; 29° N. of E.
4. 14·9 lbwt. ; 71° from the line of action of the force 10 lbwt.
5. 44·3 feet per second ; 32° N. of E.
6. 22·6 lbwt. ; 34° from the line of action of the force 11 lbwt.
7. 16·1 ; 7° S. of E.

EXAMPLES.—CXLVI.

1. 4·68 ; 13° N of E. **2.** 474 ; 29° N. of E.
3. 4·43 ; 32° N of E. **4.** 1610 ; 7° S. of E.
5. (i.) 6·06 ; 21° N. of E. (ii.) 6·03 ; 10° S. of E.
6. (i.) 652 ; 38° N. of E. (ii.) 841 ; 43° N. of E.
7. (i.) 6·89 ; 28° N. of E. (ii.) 1·13 ; due N.

Examples.—CXLVII.

1. 36·05 ; 34° N. of E. **2.** 29·16 ; 31° W. of N.
3. 8·944 ; 27° W. of S. **4.** 10·77 ; 22° S. of E.
5. See Examples.—CXLVI. **6.** 7·845 ; 10° W. of N.

Examples.—CXLVIII.

1. 36·05 lbwt., acting to the left at an angle of 34° below the horizontal.
2. 28·82 ; $2\frac{1}{2}$° N. of E. **3.** 60·21 ; 42° W. of N.
4. 4·298 miles an hour ; 17° E. of S. **5.** 14·73 miles an hour ; 16° N. of F

EXAMINATION TABLES*

A copy of these Tables is supplied to each candidate at the Examinations of the Science and Art Department, in Elementary Practical Mathematics, Applied Mechanics, and Steam.

USEFUL CONSTANTS.

1 inch = 25·4 mms.

1 gallon = ·1605 cub. foot = 10 lbs. of water at 62° F.

1 knot = 6080 feet per hour.

Weight of 1 lb. in London = 445,000 dynes.

One pound avoirdupois = 7000 grains = 453·6 grammes.

1 cub. foot of water weighs 62·3 lbs.

1 cub. foot of air at 0° C. and 1 atmosphere, weighs ·0807 lb.

1 cub. foot of hydrogen at 0° C. and 1 atmosphere, weighs ·00559 lb.

1 foot-pound = $1·3562 \times 10^7$ ergs.

1 horse-power-hour = 33,000 × 60 foot-pounds.

1 electrical unit = 1000 watt-hours.

Joule's equivalent to suit Regnault's H, is $\begin{cases} 774 \text{ foot-pounds} = 1 \text{ Fahr. unit.} \\ 1393 \text{ foot-pounds} = 1 \text{ Cent. ,,} \end{cases}$

1 horse-power = 33,000 foot-pounds per minute = 746 watts.

Volts × amperes = watts.

1 atmosphere = 14·7 lbs. per sq. inch = 2116 lbs. per sq. foot = 760 mms. of mercury = 10^6 dynes per. sq. cm. nearly.

A column of water 2·3 feet high corresponds to a pressure of 1 lb. per sq. inch.

Absolute temp., $t = \theta^\circ$ C. + 273·7°.

Regnault's H = 606·5 + ·305 θ° C. = 1082 + ·305 θ° F.

$pu^{1.0646} = 479$,

$$\log_{10} p = 6·1007 - \frac{B}{t} - \frac{C}{t^2},$$

where $\log_{10} B = 3·1812$, $\log_{10} C = 5·0881$,

p is in pounds per square inch, t is absolute temperature Centigrade

u is the volume in cubic feet per pound of steam.

$\pi = 3·1416$.

1 radian = 57·3°.

To convert common into Napierian logarithms, multiply by 2·3026.

The base of the Napierian logarithms is $e = 2·7183$.

The value of g at London = 32·182 feet per second per second.

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* By permission of the Controller of His Majesty's Stationery Office.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	25	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7338	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	3681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	3739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine			
De-grees.	Radians.								
0°	0	000	0	0	8	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1 3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1 8807	·8829	1 030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1 0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine.	Co-tangent.	Tangent.	Sine.	Chord.	Radians.	De-grees.
								Angle.	

EXAMINATION PAPER

IN

ELEMENTARY PRACTICAL MATHEMATICS.

BOARD OF EDUCATION,

May, 1899.

Questions marked with an asterisk involve methods which are not included in the revised Syllabus.

Use the Examination Tables given in the preceding pages.
Only seven questions to be answered.

1. Using either a slide rule or table of logarithms, or by contracted arithmetical methods, calculate $23{\cdot}51 \times 6{\cdot}78$, $23{\cdot}51 \div 6{\cdot}78$; $23{\cdot}51 \div 0{\cdot}0678$, $\sqrt[3]{23{\cdot}51}$, $6{\cdot}78^{2.34}$, $0{\cdot}678^{-1.301}$.

Why is it that we add logarithms of numbers to obtain the logarithm of their product? Why do we divide the logarithm of a number by 2 to obtain the logarithm of its square root? (14)

2. Answer *only one* of the following questions, (*a*), (*b*), or (*c*) :—

(*a*) In the following formula, $a = 25$, $b = 8{\cdot}432$, $c = 0{\cdot}345$, $\theta = 0{\cdot}4226$ radians : find the value of—

$$a^{1.157}\, b^{-\frac{1}{3}} \div \theta(c^3 + a \log_e b \,.\, \tan\theta) \qquad (14)$$

(*b*) Work out the values of—

$$(sr^{-1} - r^{-s}) \div (s - 1) \text{ and } (1 + \log_e r) \div r$$

when $s = 0{\cdot}95$, and $r = 1{\cdot}75$. (14)

(*c*) If $pu^{1.0646} = 479$, find p when u is $12{\cdot}12$; find u when p is $60{\cdot}4$. (14)

3. Answer *only one* of the following questions, (*a*), (*b*), or (*c*) :—

* (*a*) A cubic cm. of mercury weighs 13·6 grammes : obtain the equivalent of a pressure of 760 mms. of mercury in inches of mercury, in feet of water, in pounds per square inch and per square foot, and in kilogrammes per square cm. (14)

(*b*) A right circular cone was measured. The method of measurement was such that we only know that the diameter of base is not less than 6·22 nor more than 6·24 inches, and the slant side is not less than 9·42 nor more than 9·44 inches. Find the slant area of the cone, taking (1) the lesser dimensions, (2) the greater dimensions. Express half the difference of the two answers as a per-centage of the mean of the two.

In calculating the area, if a man gives ten significant figures in his answer, how many of these are unnecessary? (14)

(*c*) 260 feet of round copper wire weigh 30 lbs.: find its diameter if a cubic inch of the copper weighs 0·32 lb. If the same weight of the copper is shaped like a hollow cylinder, 1 inch internal diameter and 2 inches long, what is its external diameter? (14)

4. Answer *only one* of the following, (*a*) or (*b*) :—

(*a*) The tangent of an angle is 0·675 : draw the angle, without using tables, and explain your construction.

Along the lines forming the angle set off lengths OA = 4·23″ and OB = 3·76″. Find the length AB, either by measurement or by calculation. (12)

(*b*) The lengths of two sides of a triangle are 3·8 and 4·6 inches, and the angle between them is 35° ; determine by drawing, or in any way you please, (1) the length of the third side, and (2) the area of the triangle. (12)

5. Answer *only one* of the following, (*a*), (*b*), or (*c*) : –

† (*a*) By using squared paper, or by any other method, divide the number 420 into two parts such that their product is a maximum. Describe your method. (14)

(*b*) If H is proportional to $D^{\frac{2}{3}}v^3$, and if H is 871 when D is 1330 and v is 12, find H when D is 1200 and v is 15. (14)

(*c*) Calculate the ratio of d_0 to d_1 from the equation—

$$d_0^3 = 1 \cdot 3 \frac{d_0^4 - d_1^4}{d_0} \qquad (14)$$

6. Answer *only one* of the following questions, (*a*) or (*b*) :—

(*a*) A body like the trunk of a tree, 13 feet long, its axis being straight, has the following cross-sectional area of A square inches at the following distances, x inches from its end. Find its volume, using squared paper. (14)

The following table gives the value of A for each value of x :—

x	0	20	35	56	72	95	110	140	156
A	405	380	362	340	325	304	287	260	252

(*b*) The transverse sections of a vessel are 15 feet apart, and their areas in square feet up to the load water-line are 4·8, 39·4, 105·4, 159·1, 183·5, 173·3, 127·4, 57·2, and 6·0 respectively. Find the volume of water displaced by the ship between the two end sections given above. (13)

† Trace $y = x(420 - x)$, and find the maximum value of y.

***7.** Find the distance of the centre of gravity, from the end from which x is measured in (a) Question 6. (14)

8. Answer *only one* of the following, (a) or (b) :—

(a) The following values of two quantities, which we may call x and y, were measured. Thus when x was found to be 1, y was found to be 0·223 :—

x	1	1·8	2·8	3·9	5·1	6·0
y	0·223	0·327	0·525	0·730	0·910	1·095

It is known that there is a law like—

$$y = a + bx$$

connecting these quantities, but the observed values are slightly wrong. Plot the values of x and y on squared paper, and find the most likely values of a and b. (14)

(b) The following measurements were made at an electric light station under steady conditions of output :—

W is the weight in pounds of feed-water per hour, and P the electric power, in kilowatts, given out by the station. When P was 50, W was found to be 3800; and when P was 100, W was found to be 5100.

If it is known that the following law is nearly true—

$$W = a + bP$$

using squared paper or any way you please, find W when P is 70 kilowatts.

State the value of W ÷ P in each of the three cases. (14)

9. The population P (millions) of a certain country is ascertained at the beginning of each of these years to be—

Year.	1830	1835	1840	1850	1860	1865	1870	1880	1890	1895
P	20	22·1	23 5	29·0	34·2	38·2	41·0	49·4	57·7	63·3

Let t be the time in years from 1830. Show the relation of P and t by a simple curve going fairly evenly through the plotted points. Find what the population probably was at the beginning of 1848, and the rate of increase of population per annum then. (14)

***10.** Answer *only one* of the following questions, (a), (b), or (c) :—

(a) Define *curvature* and *radius of curvature.*

The length of a railway curve which has a uniform curvature is one mile, and the total change of direction is 30°: find the value of the curvature and of the radius of curvature. (13)

(*b*) Define *radian* and *angular velocity.*

A wheel makes 90 turns per minute: what is its angular velocity in radians per second? If a point on the wheel is 6 feet from the axis, what is its linear speed? (12)

(*c*) The earth being supposed spherical of 4000 miles radius, what is the linear velocity in miles per hour of a point in 36° N. latitude? The earth makes one revolution in 23·93 hours. (14)

11. Describe any system which you know of that enables us to define exactly the position of a point in space.

Work *one* but not *both* of the following, (*a*) or (*b*):—

(*a*) The three rectangular co-ordinates of a point P are 3, 4, and 5: determine (1) the length of the line joining P to O, the origin of co-ordinates; (2) the cosines of the angles which OP makes with the three rectangular axes. (14)

*(*b*) The polar co-ordinates of a point are—

$$r = 3, \theta = 65°, \phi = 50°$$

Determine its rectangular co-ordinates. (14)

12. Only *one* of the following, (*a*) or (*b*), is to be answered:—

(*a*) A straight line 4 inches long makes 30° with the horizontal plane of projection, and 50° with the vertical plane. Draw its projections on these planes; measure and write down their lengths. (14)

*(*b*) Represent by its traces a plane inclined 48° and 55° respectively to the horizontal and vertical planes of projection. Measure and write down the angles which the traces make with the line in which the planes of projection intersect each other. (14)

13. There are three vectors in one plane—

A, of amount 2 in the direction towards the north-east.
B, of the amount 3 in the direction towards the north.
C, of the amount 2·5 in the direction towards 20° east of south.

By drawing, or any methods of calculation, find the following vector sums and differences—

$$A + B + C, \quad B + C - A, \quad A - C \qquad (14)$$

14. Answer *only one* of the following questions, (*a*) or (*b*):—

(*a*) What is meant by the slope of a curve *at a point* on the curve? How is this measured? If the co-ordinates of points on the curve represent two varying quantities, say, distance and time, what does the slope of the curve at any place represent? Prove your statement. (14)

(*b*) At the end of a time t it is observed that a body has passed over a distance s reckoned from some starting-point. If it is known that—

$$s = 20 + 12t + 7t^2$$

find s when t is 5, and by taking a slightly greater value of t, say, 5·001, calculate the new value of s and find the average velocity during the 0·001 second. How would you proceed to find the exact velocity at the instant when t is 5, and how much is this velocity? (14)

15. Plot *only one* of the following curves from $x = 0$ to $x = 8$. To calculate five values of y will probably be enough :—

$$y = 4x^{0.70} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = 2{\cdot}3 \sin\left(0{\cdot}2618x + \frac{\pi}{6}\right) \quad . \quad . \quad . \quad (2)$$

$$y = 0{\cdot}53\, e^{0.26x} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also give the answer to *only one* of the following questions, (*a*) or (*b*), using any method of calculation you please :—

(*a*) In the case chosen by you find the rate of increase of y with regard to x where $x = 3$; or (*b*) find the average value of y from $x = 0$ to $x = 8$. (15)

16. Find a value of x which satisfies any one of the following equations you please. *Only one* is to be attempted :—

$$2x^{3.1} - 3x - 16 = 0.$$

$$2{\cdot}42x^3 - 3{\cdot}15 \log_e x - 20{\cdot}5 = 0.$$

$$e^x - e^{-x} + 0{\cdot}4x - 10 = 0.$$

The answer to be given correctly to *three* significant figures. (15)

RESULTS

1. 159·4 ; 3·468 ; 346·8 ; 2·865 ; 88·1 ; 1 658.

2. (*a*) 2·009 ; (*b*) 0·8948 ; 0·8911 ; (*c*) 33·63 ; 6·993.

3. (*a*) 29·92 inches of mercury ; 33·9 feet of water ; 14·67 lbs. per sq. inch ; 2113 lbs. per sq. foot ; 1·034 kilog. per sq. cm.

(*b*) 92·04 sq. inches ; 92·53 sq. inches ; 0·265 per cent. The measurements are roughly correct to the third significant figure ; therefore only the first three significant figures of the result are of any value.

(*c*) 0·1956 inch ; 7·79 inches.

4. (*a*) 2·4 inches ; (*b*) 2·64 inches, 5·02 sq. inches.

5. (*a*) 210, 210 ; (*b*) 1588 ; (*c*) 1·443 : 1.

6. (*a*) 28·82 cub. feet ; (*b*) 12,800 cub. feet. **7.** 72·1 inches.

8. (*a*) $y = 0{\cdot}031 + 0{\cdot}176x$; (*b*) 4320 lbs. per hour ; 76, 51, 61·7.

9. 28·4 millions ; 0·48 million per annum.

10. (*a*) 0·01705 degrees per yard ; 3362 yards. (*b*) 9·425 radians per second ; 56·55 feet per second. (*c*) 849·4 miles per hour.

11. (*a*) (1) 7·071 ; (2) 0·4243, 0·5657, 0·7071. (*b*) 1·747, 2·082, 1·268.

12. (*a*) 3·46 inches, 2·57 inches. (*b*) 39°, 35°.

13. 3·07, 42° north of east ; 0·95, 36° west of south ; 3·8, 8° east of north.

14. (*a*) See §§ 213, 214. (*b*) 82·007 feet per second ; 82 feet per second.

15. (*a*) (1) increase of 2·01 in y for 1 in x ; (2) increase of 0·156 in y for 1 in x ; (3) increase of 0·301 in y for 1 in x.

(*b*) (1) 10·1 ; (2) 1·9 ; (3) 2·04.

16. (1) 2·18 ; (2) 2·11 ; (3) 2·22.

EXAMINATION PAPER

IN

ELEMENTARY PRACTICAL MATHEMATICS.

BOARD OF EDUCATION,

May, 1900.

Questions marked with an asterisk involve methods which are not included in the revised Syllabus.

Use the Examination Tables given on pages 353–358.
Only seven questions to be answered.

1. Compute $168 \cdot 3 \times 2 \cdot 476$, $16 \cdot 83 \div 24 \cdot 76$, $0 \cdot 1683 \div 0 \cdot 002476$, $1 \cdot 683^{3 \cdot 65}$, $0 \cdot 01683^{-0 \cdot 26}$.

Why is it that we subtract logarithms of numbers to obtain the logarithm of their quotient?

Why do we divide the logarithm of a number by 3 to obtain the logarithm of its cube root? (14)

2. Find the value of—

$$a^{-\frac{1}{3}} (a^2 - b^2)^{\frac{1}{2}} \div \left(\sin \theta \, . \, \log_e \frac{a}{b}\right)$$

if a is 9·632, b is 2·087, θ is 0·3840 radians. (14)

3. Answer *only one* of the following questions, (a) or (b) :—

(a) A hollow cylinder is 4·32 inches long; its external and internal diameters are 3·150 and 1·724 inches: find its volume and the sum of the areas of its two curved surfaces. (13)

(b) A circular anchor ring has a volume 930 cub. inches and an area 620 sq. inches: find its dimensions. (14)

4. In a triangle ABC the angle C is 53°, the sides AC and AB are 0·523 and 0·942 mile respectively: find the side CB in miles and the area of the triangle in square miles, either by actual construction on your paper or by calculation. (13)

5. Answer *only one* of the following, (a) or (b) :—

(a) If H is proportional to $D^{\frac{2}{3}} v^3$, and if D is 1810 and v is 10 when H is 620, find H if D is 2100 and v is 13. (14)

(b) If $y = ax^{\frac{1}{2}} + bxz^2$;
If $y = 62.3$ when $x = 4$ and $z = 2$;
If $y = 187.2$ when $x = 1$ and $z = 1.46$:
find a and b; and find the value of y when x is 9 and z is 0·5. (14)

6. h is the height in feet of the atmospheric surface of the water in a reservoir above the lowest point of the bottom; A is the area of the surface in square feet.

When the reservoir was filled to various heights, the areas were measured and found to be—

Values of h	0	13	23	33	47	62	78	91	104	120
Values of A	0	21000	27500	33600	39200	44700	50400	54700	60800	69300

How many cubic feet of water leave the reservoir when h alters from 113 to 65? (14)

7. A man sells kettles. He has only made them of three sizes as yet, and he has fixed on the following as fair list prices:—

12 pint kettle, price 68 pence.
6 ,, ,, 50 ,,
2 ,, ,, 22 ,,

He knows that other sizes will be wanted, and he wishes to publish at once a price list for many sizes. State the probably correct list prices of his 4 and 8 pint kettles. (14)

8. The following corresponding values of two quantities, which we may call x and y, were measured:—

x	0·5	1·7	3·0	4·7	5·7	7·1	8·7	9·9	10·6	11·8
y	148	186	265	326	388	436	529	562	611	652

It is known that there is a law like—

$$y = a + bx$$

connecting these quantities, but the observed values are slightly wrong. Plot on squared paper; find the most probable values of a and b, and state the probable error in the measured value of y when $x = 8.7$. (14)

9. Find, correctly to three significant figures, a value of x which will satisfy this equation—

$$9x^3 - 41x^{0.8} + 0.5e^{2x} - 92 = 0 \qquad (15)$$

† **10.** Divide the number 20 into two parts, such that the square of one, together with three times the square of the other, shall be a minimum. Use any method you please. (14)

† Trace $y = x^2 + 3(20 - x)^2$, and find the minimum value of y.

11. Answer *only one* of the following, (*a*) or (*b*) :—

(*a*) The three rectangular co-ordinates of a point P are $x = 1{\cdot}5$, $y = 2{\cdot}3$, $z = 1{\cdot}8$. Find (1) the length of the line joining P to O the origin; (2) the cosines of the angles which OP makes with the three rectangular axes. (13)

*(*b*) The polar co-ordinates of a point are—

$$r = 20, \theta = 32^\circ, \phi = 70^\circ$$

Find the rectangular co-ordinates. (14)

***12.** There are three lines OX, OY, and OZ mutually at right angles. The following lengths are set off along these lines :—

OA,	of length	2 inches,	along	OX
OB,	,,	3·4 ,,	,,	OY
OC,	,,	2·95 ,,	,,	OZ

A plane passes through A, B, and C.

Determine and measure the angle between this plane and the plane which contains the lines OX and OY.

Also determine and measure the angle between the plane and the line OZ. (14)

13. Answer *only one* of the following questions, (*a*) or (*b*) :—

There are three vectors in a horizontal plane—

A, of amount 1·5 towards the south-east.
B, ,, 3·9 in the direction towards 20° west of south.
C, ,, 2·7 towards the north.

(*a*) Find the vector sums or differences—

$$A + B + C, \quad A - B + C, \quad B - C \qquad (14)$$

*(*b*) Find the scalar products A.B and A.C. (15)

14. Answer *only one* of the following, (*a*), (*b*), or (*c*) :—

(*a*) What is meant by the *slope* of a curve *at a point* on the curve? How is this measured? If the co-ordinates of points on the curve represent two varying quantities, say, *distance* passed through by a body and the *time* that has elapsed, what does the slope of the curve at any place represent? Prove your statement. (14)

(*b*) † A certain quantity y depends upon x in such a way that—

$$y = a + bx + cx^2$$

where a, b, and c are given constant numbers. Prove that the rate of increase of y with regard to x is—

$$b + 2cx \qquad (14)$$

*(*c*) ‡ If there are two curves, one showing how y depends upon x; the other how z depends upon x: If for every value of x the slope of the second curve is proportional to the ordinate of the first, show that z represents the area of the y curve. (14)

---

† Use the method of § 216.

‡ Prove that the rate of increase of the area is measured by the ordinate.

RESULTS

1. 416·8 ; 0·6797 ; 67·97 ; 6·688 ; 2·892. **2.** 7·714.

3. (*a*) 23·58 cub. inches ; 66·15 sq. inches.

(*b*) mean radius = 5·236 inches ; radius of circular section = 3 inches.

4. 1·16 miles ; 0·242 sq. mile.

5. (*a*) 1503. (*b*) $y = 243{\cdot}9x^{\frac{1}{2}} - 26{\cdot}59xz^2$; 671·9.

6. 2,635,000 cub. feet. **7.** $37\frac{1}{2}d.$, $59\frac{1}{4}d.$

8. $y = 119 + 45{\cdot}7x$; 516·6 instead of 529. **9.** 2·36. **10.** 15, 5.

11. (*a*) 3·28 inches ; 0·4568 ; 0·7004 ; 0·5483. (*b*) 3·624 ; 9·959 ; 16·96.

12. 60°, 30°.

13. (*a*) 2·04, 8° west of south ; 5·82, 24° east of north ; 6·5, 12° west of south.

(*b*) 2·47 ; − 2·86. **14.** (*a*) See §§ 213, 214.

EXAMINATION PAPER

IN

ELEMENTARY PRACTICAL MATHEMATICS.

BOARD OF EDUCATION,

May, 1901.

Use the Examination Tables given on pages 353–358.
Only eight questions to be answered.

1. Compute $30 \cdot 56 \div 4 \cdot 105$, $0 \cdot 03056 \times 0 \cdot 4105$, $4 \cdot 105^{1 \cdot 23}$, $\cdot 04105^{-2 \cdot 3}$.
The answers must be right to three significant figures.
Why do we multiply log a by b to obtain the logarithm of a^b? (10)

2. Answer *only one* of the following (a) or (b) :—

(a) Find the value of—

$$ae^{-bt} \sin (ct + g)$$

if $a = 5$, $b = 200$, $c = 600$, $g = -0 \cdot 1745$ radian, $t = \cdot 001$.
(Of course the angle is in radians.) (10)

(b) Find the value of—

$$\sin A \cos B - \cos A \sin B$$

if A is 65° and B is 34°. (8)

3. A tube of copper (0·32 lb. per cubic inch) is 12 feet long, and of 3 inches inside diameter; it weighs 100 lbs. Find its outer diameter, and the area of its curved outer surface. (10)

4. ABC is a triangle. The angle A is 37°, the angle C is 90°, and the side AC is 5·32 inches; find the other sides, the angle B, and the area of the triangle. (10)

5. An army of 5000 men costs a country £800,000 per annum to maintain it; an army of 10,000 men costs £1,300,000 per annum to maintain it; what is the annual cost of an army of 8000 men? Take the simplest law which is consistent with the figures given. Use squared paper or not, as you please. (12)

6. In any class of turbine if P is the power of the waterfall and H the height of the fall, and n the rate of revolution, then it is known that for any particular class of turbines of all sizes—

$$n \propto H^{1 \cdot 25} P^{-0 \cdot 5}$$

In the list of a particular maker I take a turbine at random for a fall of 6 feet, 100 horse-power, 50 revolutions per minute. By means of this I find I can calculate n for all the other turbines of the list. Find n for a fall of 20 feet and 75 horse-power. (12)

7. At the following draughts in sea water a particular vessel has the following displacements :—

Draught h feet	15	12	9	6·3
Displacement T tons	2098	1512	1018	586

What are the probable displacements when the draughts are 11 and 13 feet respectively? (10)

8. The three parts (a), (b), (c), must all be answered to get full marks.

(a) There are two quantities, a and b. The square of a is to be multiplied by the sum of the squares of a and b; add 3; extract the cube root; divide by the product of a and the square root of b. Write down this algebraically.

(b) Express $\frac{1}{x^2 - 7x + 12}$ as the sum of two simpler fractions.

(c) A crew which can pull at the rate of six miles an hour, finds that it takes twice as long to come up a river as to go down; at what rate does the river flow? (14)

9. A number is added to 2·25 times its reciprocal; for what number is this a minimum? Use squared paper or the calculus, as you please. (12)

10. If $y = \frac{1}{2}x^2 - 3x + 3$, show, by taking some values of x, and calculating y, and plotting on squared paper, the nature of the relationship between x and y. For what values of x is $y = 0$? (14)

11. The keeper of a restaurant finds that when he has G guests a day, his total daily profit (the difference between his actual receipts and expenditure including rent, taxes, wages, wear and tear, food and drink) is P pounds; the following numbers being averages obtained by comparison of many days' accounts, what simple law seems to connect P and G?

G.	P.
210	− 0·9
270	+ 1·8
320	+ 4·8
360	+ 6·4

For what number of guests would he just have no profit? (12)

12. At the end of a time t seconds it is observed that a body has passed over a distance s feet reckoned from some starting point. If it is known that—

$$s = 25 + 150\,t - 5t^2$$

what is the velocity at the time t?

Prove the rule that you adopt to be correct. If corresponding values of s and t are plotted on squared paper, what indicates the velocity, and why? (14)

13. The three rectangular co-ordinates of a point P are 2·5, 3·1, and 4. Find (1) the length of the line joining P with O the origin; (2) the cosines of the angles which OP makes with the three axes; and (3) the sum of the squares of the three cosines. (14)

RESULTS

1. 7·445; ·01254; 5·680; 1547. **2.** (*a*) 1·69. (*b*) ·515.
3. 3·43 inches; 1550 sq. inches.
4. $a = 4·01$ inches; $c = 6·66$ inches; $B = 53°$; area = 10·7 sq. inches.
5. £1,100,000. **6.** 260 revolutions per minute.
7. 1375 tons; 1720 tons.
8. (*a*) $\dfrac{\{a^2(a^2+b^2)+3\}^{\frac{1}{3}}}{ab^{\frac{1}{2}}}$. (*b*) $\dfrac{1}{x-4} + \dfrac{1}{3-x}$. (*c*) 2 miles an hour.
9. 1·5. **10.** 4·73; 1·27. **11.** P = ·0515 G − 11·91; 231.
12. $150 - 10t$. **13.** (1) 5·644. (2) ·4429, ·5492, ·7087. (3) 1.

EXAMINATION PAPER

IN

ELEMENTARY PRACTICAL MATHEMATICS.

BOARD OF EDUCATION,

May, 1902.

Use the Examination Tables given on pages 353–358.

Only eight questions to be answered. Three of these must be Nos. 1, 2, and 3.

1. Compute by contracted methods without using logarithms $23{\cdot}07 \times 0{\cdot}1354$, $2307 \div 1{\cdot}354$.

Compute $2{\cdot}307^{0{\cdot}65}$ and $23{\cdot}07^{-1{\cdot}25}$, using logarithms.

The answers to consist of four significant figures.

Why do we add logarithms to obtain the logarithm of a product? (10)

2. Answer only *one* of the following (*a*) or (*b*):—

(*a*) If—

$$w = 144\{p_1(1 + \log r) - r(p_3 + 10)\}$$

and if $p_1 = 100$, $p_3 = 17$; find w for the four values of r, $1\frac{1}{2}$, 2, 3, 4. Tabulate your answers.

(*b*) If c is 20 feet, $D = 6$ feet, $d = 3$ feet, find θ in radians if—

$$\sin\theta = \frac{D + d}{2c}$$

Now calculate L, the length of a belt, if—

$$L = (D + d)\left(\frac{\pi}{2} + \theta + \frac{1}{\tan\theta}\right) \qquad (10)$$

3. The three parts (*a*), (*b*), and (*c*) must all be answered to get full marks.

(*a*) Let x be multiplied by the square of y, and subtracted from the cube of z; the cube root of the whole is taken and is then squared. This is divided by the sum of x, y, and z. Write all this down algebraically.

(*b*) Express—

$$\frac{x - 13}{x^2 - 2x - 15}$$

as the sum of two simpler fractions.

(*c*) The sum of two numbers is 76, and their difference is equal to one-third of the greater: find them. (10)

4. What is the idea on which compound interest is calculated? Explain, as if to a beginner, how it is that—

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where P is the money lent, and A is what it amounts to in n years at r per cent. per annum.

If A is 130, and P is 100, and n is 7·5, find r. (14)

5. Suppose s the distance in feet passed through by a body in the time seconds is $s = 10t^2$. Find s when t is 2, find s when t is 2·01, and also when t is 2·001. What is the average speed in each of the two short intervals of time after $t = 2$? When the interval of time is made shorter and shorter, what does the average speed approximate to? (14)

6. If $z = ax - by^3x^{\frac{1}{2}}$.

If $z = 1·32$ when $x = 1$ and $y = 2$;
and if $z = 8·58$ when $x = 4$ and $y = 1$;
find a and b.

Then find z when $x = 2$ and $y = 0$. (10)

7. A prism has a cross-section of 50·32 square inches. There is a section making an angle of 20° with the cross-section: what is its area? Prove the rule that you use. (10)

8. In a triangle ABC, AD is the perpendicular on BC; AB is 3·25 feet; the angle B is 55°. Find the length of AD. If BC is 4·67 feet, what is the area of the triangle?

Find also BD and DC and AC. Your answers must be right to three significant figures. (10)

9. It is known that the weight of coal in tons consumed per hour in a certain vessel is $0·3 + 0·001v^3$, where v is the speed in knots (or nautical miles per hour). For a voyage of 1000 nautical miles, tabulate the time in hours and the total coal consumption for various values of v. If the wages, interest on cost of vessel, etc., are represented by the value of 1 ton of coal per hour, tabulate for each value of v the total cost, stating it in the value of tons of coal, and plot on squared paper. About what value of v gives greatest economy? (14)

10. An examiner has given marks to papers; the highest number of marks is 185, the lowest 42. He desires to change all his marks according to a linear law, converting the highest number of marks into 250 and the lowest into 100; show how he may do this, and state the converted marks for papers already marked 60, 100, 150. Use squared paper, or mere algebra, as you please. (10)

11. A is the horizontal sectional area of a vessel in square feet at the water-level, h being the vertical draught in feet.

A	14,850	14,400	13,780	13,150
h	23·6	20·35	17·1	14·6

Plot on squared paper and read off and tabulate A for values of h, 23, 20, 16.

If the vessel changes in draught from 20·5 to 19·5, what is the diminution of its displacement in cubic feet? (10)

12. Find a value of x which satisfies the equation—

$$x^2 - 5 \log_{10} x - 2{\cdot}531 = 0 \qquad (14)$$

13. If $x = a(\phi - \sin \phi)$ and $y = a(1 - \cos \phi)$, and if $a = 5$; taking various values of ϕ between 0 and, say 1·5, calculate x and y, and plot this part of the curve. (14)

RESULTS

1. 3·124, 1704; 1·722, 0·01978.

2. (*a*) When r is 1½; 2; 3; 4;
value of expression is 11,104; 10,959; 9607; 7518.
(*b*) $\theta = 0{\cdot}2269$, L = 55·16.

3. (*a*) $\dfrac{(z^3 - xy^2)^{\frac{2}{3}}}{x+y+z}$; (*b*) $\dfrac{2}{x+3} - \dfrac{1}{x-5}$; (*c*) 45·6, 30·4.

4. 3·5.

5. When $t = 2$, $s = 40$; when $t = 2{\cdot}01$, $s = 40{\cdot}401$; when $t = 2{\cdot}001$, $s = 40{\cdot}04001$. Between $t = 2$ and $t = 2{\cdot}01$, average speed is 40·1 feet per second; between $t = 2$ and $t = 2{\cdot}001$, average speed is 40·01 feet per second. When $t = 2$, actual speed is 40 feet per second.

6. $a = 2{\cdot}2$, $b = 0{\cdot}11$; $z = 4{\cdot}4$.

7. 53·55 square inches.

8. AD = 2·66 feet, area = 6·22 sq. feet; BD = 1·86 feet, DC = 2·81 feet, AC = 3·87 feet.

9. 8·7 knots.

10. 119, 161, 213.

11. 14,760, 14,350, 13,530 sq. feet; 14,350 cubic feet.

12. 2·013.

13. Giving to ϕ the values 0, 0·5, 1, 1·5, we obtain for x the values 0, 0·1, 0·8, 2·5, and for y the values 0, 0·6, 2·3, 4·65.

EXAMINATION PAPER

IN

ELEMENTARY PRACTICAL MATHEMATICS.

BOARD OF EDUCATION,

May, 1903.

Use the Examination Tables given on pages 353–358.

Only eight questions to be answered. Three of these must be Nos. 1, 2, and 3.

1. Compute by contracted methods to four significant figures only, and without using logarithms—

$$8{\cdot}102 \times 35{\cdot}14, \quad 254{\cdot}3 \div 0{\cdot}09027.$$

Compute, using logarithms,

$$\sqrt[3]{37{\cdot}24}, \quad \sqrt[2]{3{\cdot}724}, \quad 372{\cdot}4^{2{\cdot}43}, \quad 0{\cdot}3724^{-2{\cdot}43}.$$

What is the theory underlying the use of logarithms in helping us to multiply, divide, and raise a number to any power? (10)

2. Answer only *one* of the following (*a*), (*b*), or (*c*):—

(*a*) If $x = \tan\theta \div \tan(\theta + \phi)$, where ϕ is always 10°, find x when θ has the values 30°, 40°, 50°, 60°, and plot the values of x and of θ on squared paper. About what value of θ seems to give the largest value of x? (10)

(*b*) At speeds greater than the velocity of sound, the air resistance to the motion of a projectile of the usual shape of weight w lbs., diameter d inches, is such that when the speed diminishes from v_1 feet per second to v, if t is the time in seconds and s is the space passed over in feet—

$$t = 7000\frac{w}{d^2}\left(\frac{1}{v} - \frac{1}{v_1}\right)$$

$$s = 7000\frac{w}{d^2}\log_e\frac{v_1}{v}$$

If v_1 is 2000, find s and t when $v = 1500$ for a projectile of 12 lbs. whose diameter is 3 inches. (10)

(*c*) Find the value of—

$$t_1 - t_3 - t_3\log_e\frac{t_1}{t_3} + l_1\left(1 - \frac{t_3}{t_1}\right)$$

if $t_1 = 458, \quad t_3 = 373, \quad l_1 = 796 - 0{\cdot}695t_1$ (10)

3. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks.

(*a*) Write down algebraically—Add twice the square root of the cube of x to the product of y squared and the cube root of z. Divide by the sum of x and the square root of y. Add four and extract the square root of the whole.

(*b*) Express—

$$\frac{3x - 2}{x^2 - 3x - 4}$$

as the sum of two simpler fractions.

(*c*) Find two numbers such that if four times the first be added to two and a half times the second the sum is 17·3, and if three times the second be subtracted from twice the first the difference is 1·2.

(*d*) In a triangle ABC, C being a right angle, AB is 14·85 inches, AC is 8·32 inches. Compute the angle A in degrees, using your tables. (10)

4. The following are the areas of cross-section of a body at right angles to its straight axis :—

A in square inches .	250	292	310	273	215	180	135	120
x inches from one end	0	22	41	70	84	102	130	145

Plot A and x on squared paper. What is the probable cross-section at $x = 50$? What is the average cross-section and the whole volume? (10)

5. The following table records the heights in inches of a girl A (born January, 1890) and a boy B (born May, 1894). Plot these records. The intervals of time may be taken as exactly four months.

Year.	1900.	1901.			1902.			1903.
Month.	Sept.	Jan.	May.	Sept.	Jan.	May.	Sept.	Jan.
A.	54·8	55·6	56·6	58·0	59·2	60·2	60·9	61·3
B.	48·3	49·0	49·8	50·6	51·5	52·3	53·1	53·9

Find in inches per year the *average* rates of growth of A and B during the given period. At about what age was the growth of A most rapid? State this rate; divide it by her average rate. (14)

6. In any such question as Question 5, where points on a curve have co-ordinates like h (height) and t (time), show exactly how it is that the slope of the curve at a point represents there the rate of growth of h as t increases. (10)

7. Find accurately to three significant figures a value of x which satisfies the equation—

$$2x^2 - 10 \log_{10} x - 3{\cdot}25 = 0 \qquad (14)$$

8. Answer only *one* of the following (a) or (b) :—

(a) A cast-iron flywheel rim (0·26 lb. per cubic inch) weighs 13,700 lbs. The rim is of rectangular section, thickness radially x, size the other way $1{\cdot}6x$. The inside radius of the rim is $14x$. Find the actual sizes. (10)

(b) The electrical resistance of copper wire is proportional to its length divided by its cross-section. Show that the resistance of a pound of wire of circular section all in one length is inversely proportional to the fourth power of the diameter of the wire. (10)

9. It is thought that the following observed quantities, in which there are probably errors of observation, follow a law like—

$$y = ae^{bx}$$

Test if this is so, and find the most probable values of a and b.

x	2·30	3·10	4·00	4·92	5·91	7·20
y	33·0	39·1	50 3	67·2	85·6	125·0

(14)

10. Plot $3y = 4{\cdot}8x + 0{\cdot}9$.
Plot $y = 2{\cdot}24 - 0{\cdot}7x$.

Find the point where they cross. What angle does each of them make with the axis of x? At what angle do they meet? (14)

11. A firm is satisfied from its past experience and study that its expenditure per week in pounds is

$$120 + 3{\cdot}2x + \frac{C}{x+5} + 0{\cdot}01C$$

where x is the number of horses employed by the firm, and C is the usual turnover.

If C is £2150, find for various values of x what is the weekly expenditure, and plot on squared paper to find the number of horses which will cause the expenditure to be a minimum. (10)

12. Assuming the earth to be a sphere, if its circumference is 360×60 nautical miles, what is the circumference of the parallel of latitude 56°? What is the length there of a degree of longitude? If a small map is to be drawn in this latitude, with north and south and east and west distances to the same scale, and if a degree of latitude (which is of course 60 miles) is shown as 10 inches, what distance will represent a degree of longitude? (14)

13. At a certain place where all the months of the year are assumed to be of the same length (30·44 days each), at the same time in each month the length of the day (interval from sunrise to sunset in hours) was measured, as in this table—

Nov.	Dec.	Jan.	Feb.	Mar.	April.	May.	June.	July.
8·35	7·78	8·35	9·87	12	14·11	15·65	16·22	15·65

What is the average increase of the length of the day (state in decimals of an hour per day) from the shortest day, which is 7·78 hours, to the longest, which is 16·22 hours? When is the increase of the day most rapid, and what is it? (14)

14. At an electricity works, where new plant has been judiciously added, if W is the annual works cost in millions of pence, and T is the annual total cost, and U the number of millions of electrical units sold, the following results have been found:—

U	W	T
0·3	0·47	0·78
1·2	1·03	1·64
2·3	1·70	2·73
3·4	2·32	3·77

Find approximately the rule connecting T and W with U. Also find the probable values of W and T when U becomes 5, if there is the same judicious management. (10)

RESULTS

1. 284·7, 2817; 3·339, 1·930, 1,768,000, 1·103.

2. (*a*) 0·6881, 0·7041, 0·6881, 0·6305, 40°; (*b*) 2684 feet, 1·5556 seconds; (*c*) 97·04.

3. (*a*) $\left\{\frac{2x^{\frac{3}{2}} + y^2 z^{\frac{1}{3}}}{x + y^{\frac{1}{2}}} + 4\right\}^{\frac{1}{2}}$. (*b*) $\frac{2}{x-4} + \frac{1}{x+1}$. (*c*) 3·229, 1·753. (*d*) 56°.

4. 312 square inches; 232 square inches, 33,650 cubic inches.

5. 2·786 inches per year, 2·4 inches per year ; $11\frac{1}{2}$ years of age ; 4·3 inches per year, 1·54.

7. 1·645. **8.** (*a*) $x = 7{\cdot}124$ inches. **9.** $a = 16{\cdot}8$, $b = 0{\cdot}28$.

10. (0·85, 1·65) ; 58°, −35° ; 87°. **11.** 21 horses.

12. 12,080 nautical miles ; 33·55 nautical miles ; 5·592 inches.

13. 0·0462 hour per day ; March, 0·073 hour per day.

14. $T = 0{\cdot}97U + 0{\cdot}48$, $W = 0{\cdot}6U + 0{\cdot}3$; 3·3, 5·33.

EXAMINATION PAPER

IN

FIRST STAGE PRACTICAL MATHEMATICS.

BOARD OF EDUCATION,

May, 1904.

Use the Examination Tables given on pages 353–358.
Answer questions 1, 2, 3, and five others.

1. The three parts (*a*), (*b*), and (*c*) must be answered to get full marks.

(*a*) Compute by contracted methods to four significant figures only, and without using logarithms

$$3{\cdot}405 \times 9{\cdot}123 \text{ and } 3{\cdot}405 \div 9{\cdot}123.$$

(*b*) Compute, using logarithms,

$$\sqrt[3]{2{\cdot}354 \times 1{\cdot}607} \text{ and } (32{\cdot}15)^{1{\cdot}52}.$$

(*c*) Write down the values of sin 23°, tan 53°, $\log_{10} 153{\cdot}4$, $\log_e 153{\cdot}4$. (10)

2. Both (*a*) and (*b*) must be answered to get full marks.

(*a*) If—

$$F = EI\pi^2 \div 4l^2,$$

If—

$$I = bt^3 \div 12,$$

If—

$$E = 3 \times 10^7,\ \pi = 3{\cdot}142,\ l = 62,\ b = 2,\ t = 0{\cdot}5,$$

find F.

(*b*) Two men measure a rectangular box; one finds its length, breadth, and depth in inches to be 5·32, 4·15, 3·29. The other finds them to be 5·35, 4·17, 3·33. Calculate the volume in each case; what is the mean of the two, what is the percentage difference of either from the mean? (10)

3. All of these (*a*), (*b*), and (*c*) must be answered to get full marks.

(*a*) Write down algebraically :—Square *a*, divide by the square of *b*, add 1, extract the square root, multiply by *w*, divide by the square of *n*.

(*b*) The ages of a man and his wife added together amount to 72·36 years; fifteen years ago the man's age was 2·3 times that of his wife: what are their ages now?

(*c*) ABC is a triangle, C being a right angle. The side AB is 15·34 inches, the side BC is 10·15 inches. What is the length of AC? Express the angles A and B in degrees. What is the area of the triangle in square inches? If this is the shape of a piece of sheet brass 0·13 inch thick, and if brass weighs 0·3 lb. per cubic inch, what is its weight? (10)

4. If—

$$y = 3x^2 - 20 \log_{10} x - 7{\cdot}077$$

find the values of y when x is 1·5, 2, 2·3. Plot the values of y and x on squared paper, and draw the probable curve in which these points lie. State approximately what value of x would cause y to be 0. (14)

5. It has been found that if P is the horse-power wasted in air friction when a disc d feet in diameter is revolving at n revolutions per minute

$$P = cd^{5{\cdot}5}\, n^{3{\cdot}5}.$$

If P is 0·1 when $d = 4$ and $n = 500$, find the constant c. Now find P when d is 9 and n is 400. (14)

6. There is a district in which the surface of the ground may be regarded as a sloping plane; its actual area is 3·246 square miles; it is shown on the map as an area of 2·875 square miles: at what angle is it inclined to the horizontal?

There is a straight line 20·17 feet long which makes an angle of 52° with the horizontal plane: what is the length of its projection on the horizontal plane? (10)

7. A British man or woman of age x years may on the average expect to live for an additional y years.

Age x.	Expected further Life y.	
	Man.	Woman.
70	8·27	8·95
60	13·14	14·24
50	18·93	20·68
40	25·30	27·46
30	32·10	34·41

Plot a curve for men and one for women, and find the expectations of life for a man and for a woman aged 54 years. (10)

8. The following tests were made upon a condensing-steam-turbine-electric-generator. There are probably some errors of observation, as the measurement of the steam is troublesome. The figures are given just as they were published in a newspaper.

Output in Kilowatts K	1,190	995	745	498	247	0
Weight W lbs. of steam consumed per hour	23,120	20,040	16,630	12,560	8,320	4,065

Plot on squared paper. Find if there is a simple approximate law connecting K and W, but do not state it algebraically. What are the probable values of K when W is 22,000 and when W is 6000? (14)

9. If—

$$y = 2x + \frac{1\cdot5}{x}$$

for various values of x, calculate y; plot on squared paper; state approximately the value of x which causes y to be of its smallest value. (14)

10. A series of soundings taken across a river channel is given by the following table, x feet being distance from one shore, and y feet the corresponding depth. Draw the section. Find its area.

x	0	10	16	23	30	38	43	50	55	60	70	75	80
y	5	10	13	14	15	16	14	12	8	6	4	3	0

(10)

11. The value of a ruby is said to be proportional to the $1\frac{1}{2}$ power of its weight. If one ruby is exactly of the same shape as another, but of 2·20 times its linear dimensions, of how many times the value is it?

[Note that the weights of similar things made of the same stuff are as the cubes of their linear dimensions.] (10)

12. x and t are the distance in miles and the time in hours of a train from a railway station. Plot on squared paper. State how the curve shows where the speed is greatest and where it is least. What is the average speed in miles per hour during the whole time tabulated?

t	0	0·05	0·10	0·15	0·20	0·25	0·30	0·35	0·40	0·45	0·50
x	0	0·25	1·00	3·05	5·00	5·85	6·10	6·10	6·35	7·00	7·65

(14)

RESULTS

1. (*a*) 31·06, 0·3732 ; (*b*) 1·558, 195·4 ; (*c*) 0·3907, 1·3270, 2·1858, 5·0330.

2. (*a*) 401 ; (*b*) 72·63 cubic inches, 74·29 cubic inches, 73·46 cubic inches, 1·13 per cent.

3. (*a*)

$$\frac{w\sqrt{\left(\frac{a^2}{b^2}+1\right)}}{n^2};$$

(*b*) 44·52, 27·84 ; (*c*) AC = 11·5 inches, A = $41\frac{1}{2}°$, B = $48\frac{1}{2}°$; area = 58·36 square inches, weight = 2·276 lbs.

4. $-3·849$, $-1·097$, 1·559 ; 2·133.

5. $1·746 \times 10^{-14}$; 3·96.

6. $27\frac{1}{2}°$; 12·42 feet.

7. 16·4 years ; 18 years.

8. Apparently a linear law ; 110 kilowatts, 1100 kilowatts.

9. When $x = 0·87$, $y = 3·46$.

10. 790 square feet.

11. 34·73.

12. About 46 miles per hour after 0·15 hour. At rest between 0·3 hour and 0·35 hour. Average velocity 15·3 miles per hour.

EXAMINATION PAPER

IN

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BOARD OF EDUCATION,

May, 1905.

Use the Examination Tables given on pages 353–358.
Answer questions 1, 2, 3, and five others.

1. The three parts (*a*), (*b*), and (*c*) must all be answered to get full marks.

(*a*) Compute by contracted methods to four significant figures only, and without using logarithms

$$12{\cdot}39 \times 5{\cdot}024 \text{ and } 5{\cdot}024 \div 12{\cdot}39.$$

(*b*) Compute, using logarithms,

$$\sqrt[2]{2{\cdot}607} \text{ and } 26{\cdot}07^{1{\cdot}13}.$$

(*c*) Write down the values of cos 35°, tan 52°, $\sin^{-1} 0{\cdot}4226$, $\log_{10} 14{\cdot}36$, $\log_e 14{\cdot}36$. (10).

[Note. $\sin^{-1} n$ means the angle whose sine is n.]

2. The three parts (*a*), (*b*), and (*c*) must all be answered to get full marks.

(*a*) If—

$$x = a(\phi - \sin\phi) \text{ and } y = a(1 - \cos\phi),$$

find

$$x \text{ and } y \text{ when } a \text{ is } 10 \text{ and } \phi = 0{\cdot}5061 \text{ radian.}$$

(*b*) In a piece of coal there was found to be 11·30 lbs. of carbon, 0·92 lb. of hydrogen, 0·84 lb. of oxygen, 0·56 lb. of nitrogen, 0·71 lb. of ash. There being nothing else, state the percentage composition of the coal.

(*c*) A brass tube, 8 feet long, has an outside diameter 3 inches, inside 2·8 inches. What is the volume of the brass in cubic inches? If a cubic inch of brass weighs 0·3 lb., what is the weight of the tube? (10)

3. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks.

(*a*) Write down algebraically:—Three times the square of x, multiplied by the square root of y; from this subtract a times the Napierean logarithm of x; again, subtract b times the sine of cx; divide the result by the sum of the cube of x and the square of y.

(*b*) Express—

$$\frac{3x + 5}{x^2 + x - 12}$$

as the sum of two simpler fractions.

(*c*) Some men agree to pay equally for the use of a boat, and each pays 15 pence. If there had been two more men in the party, each would have paid 10 pence. How many men were there, and how much was the hire of the boat?

(*d*) The altitude of a tower observed from a point distant 150 feet horizontally from its foot is 26°: find its height. (10)

4.

$$\text{If} \quad p_1 v_1^{1\cdot13} = p_2 v_2^{1\cdot13}$$
$$\text{and if} \quad v_2/v_1 \text{ be called } r;$$
$$\text{if} \quad p_2 = 6,$$
$$\text{find } r \text{ if } p_1 = 150.$$

(14)

5.

$$\text{If} \quad y = \frac{2}{x} + 5 \log_{10} x - 2\cdot70,$$

find the values of y when x has the values 2, 2·5, 3.

Plot the values of y and x on squared paper, and draw the probable curve in which these points lie. State approximately what value of x would cause y to be 0. (14)

6. x and t are the distance in miles and the time in hours of a train from a railway station. Plot on squared paper. Describe why it is that the *slope* of the curve shows the speed; where is the speed greatest and where is it least?

x	0	0·12	0·50	1·52	2·50	2·92	3·05	3·17	3·50	3·82	4·15
t	0·00	0·05	0·10	0·15	0·20	0·25	0·30	0·35	0·40	0·45	0·50

(14)

7. A vessel is shaped like the frustum of a cone, the circular base is 10 inches diameter, the top is 5 inches diameter, the vertical axial height is 8 inches. By drawing, find the axial height to the imaginary vertex of the cone. If x is the height of the surface of a liquid from the bottom, plot a curve, to any scales you please, showing for any value of x the area of the horizontal section there. Three points of the curve will be enough to find. (14)

8. A circle is 3 inches diameter, its centre is 4 inches from a line in its plane. The circle revolves about the line as an axis and so generates a ring. Find the volume of the ring, also its surface area. (10)

9. If u is usefulness of flywheels,

$$u \propto d^5 n^2,$$

if d is the linear size (say diameter) and n the speed. We assume all flywheels to be similar in shape. I wish to have the usefulness one hundred times as great, the speed being trebled: what is the ratio of the new diameter to the old one? (14)

10. The total cost C of a ship per hour (including interest and depreciation on capital, wages, coal, &c.) is $C = a + bs^3$, where s is the speed in knots (or nautical miles per hour).

When s is 10, C is found to be £5·20.
When s is 15, C is found to be £7·375.

Calculate a and b. What is C when s is 12?

How many hours are spent in a passage of 3000 nautical miles at a speed of 12 knots, and what is the total cost of the passage? (14)

11. A feed-pump of variable stroke was driven by an electromotor at constant speed; the following experimental results were obtained:—

Electrical Horse-power.	Power given to Water.
3·12	1·19
4·5	2·21
7·5	4·26
10·74	6·44

Plot on squared paper, and state the probable electrical power when the power given to the water was 5. (10)

12. Mr. Scott Russell found that at the following speeds of a canal boat the tow-rope pull was as follows:—

Speed in miles per hour . .	6·19	7·57	8·52	9·04
Tow-rope pull in pounds . .	250	500	400	280

What was the probable pull when the speed was 8 miles per hour? There was reason to believe that the pull was at its maximum at 8 miles per hour, because this was the natural speed of a long wave in that canal. (14)

RESULTS

1. (*a*) 62·25, 0·4055; (*b*) 1·614, 39·83; (*c*) 0·8192, 1·2799, 25°, 1·1571, 2·6643.

2. (*a*) 0·213, 1·254; (*b*) carbon, 78·86 per cent.; hydrogen, 6·42 per cent.; oxygen, 5·86 per cent.; nitrogen, 3·91 per cent.; ash, 4·95 per cent.; (*c*) 87·46 cubic inches, 26·24 lbs.

3. (*a*) $\dfrac{3x^2\sqrt{y} - a\log_e x - b\sin cx}{x^3 + y^2}$; (*b*) $\dfrac{1}{x+4} + \dfrac{2}{x-3}$; (*c*) 4 men, 5*s.*; (*d*) 73·15 ft.

4. 17·26.

5. $-0·1950$, $+0·0895$, $+0·3522$; $x = 2·34$.

6. Apparently about 30 miles per hour after 0·15 hour; reduced nearly to rest [say ¼ mile per hour] after 0·3 hour.

7. Height of vertex = 16 inches.

8. 177·6 cubic inches; 236·8 square inches.

9. 1·618.

10. $a = 4·284$, $b = 0·000916$; £5·867; 250 hours, £1467.

11. 8·61.

12. 510 lbs.

EXAMINATION PAPER

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BOARD OF EDUCATION,

May, 1906.

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Answer questions 1, 2, 3, and five others.

1. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks.

(*a*) Compute by contracted methods, to four significant figures only and without using logarithms

$$3{\cdot}214 \times 0{\cdot}7423 \div 7{\cdot}912.$$

(*b*) Write down the logarithms of 32170, 32·17, 0·3217, 0·003217.

(*c*) Compute, using logarithms

$$\sqrt{84{\cdot}05 \times 0{\cdot}1357 \div 1{\cdot}163}.$$

(*d*) Express £0 17*s.* 9*d.* as the decimal of a pound. (10)

2. The three parts (*a*), (*b*), and (*c*) must all be answered to get full marks.

(*a*) If $A = P\left(1 + \frac{r}{100}\right)^n$, find A when $P = 200$, $r = 4$, $n = 12$.

(*b*) On board a ship there were 1312 men, 514 women, and 132 children. State these as percentages of the total number of persons.

(*c*) When x and y are small we may take

$\frac{1+x}{1+y}$ as being very nearly equal to $1 + x - y$.

What is the error in this when $x = 0{\cdot}02$ and $y = 0{\cdot}03$? (10)

3. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks.

(*a*) Write down algebraically:—The principal P multiplied by $1 + \frac{r}{100}$ twelve times.

(*b*) Divide 17·24 into two parts such that one-quarter of the first added to one-third of the second makes 5·06.

(*c*) What are the factors of $x^2 - 10$?

(*d*) A wheel is 3·45 feet in diameter; it makes 1020 revolutions rolling along a road: what is the distance passed over? (10)

4 If $y = x^2 - 3{\cdot}4x + 2{\cdot}73$, calculate y when x has the values 1, 1·2, 1·4, 1·6, 1·8, 2, and 2·2. Plot these values of x and y and draw a curve. What values of x cause y to be 0? (14)

5. x and t are the distance in miles and the time in hours of a train from a railway station. Plot on squared paper. Describe why it is that the *slope* of the curve shows the speed; where approximately is the speed greatest and where is it least?

x	0	1·5	6·0	14·0	19·0	21·0	21·5	21·8	23·0	24·7	26·8
t	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0

(14)

6. The point B is 4 miles north and 2 miles east of the point A. What is the distance from A to B, and what angle does the line AB make with the due east direction? (14)

7. The horse-power of the engines of a ship being proportional to the cube of the speed, if the horse-power is 2000 at a speed of 10 knots, what is the power when the speed is 15 knots? (10)

8. There are two maps, one to the scale of 2 inches to the mile, the other to the scale of half an inch to the mile. The area of an estate on the first map is 1·46 square inches: what is the area of this estate on the second map? (14)

9.
$$y = ax^2 + bx^3;$$

when $x = 1$, y is 4·3, and when $x = 2$, y is 30: find a and b. What is y when x is 1·5? (14)

10. In the following table A is the area in square feet of the horizontal section of a ship at the level of the surface of the water when the vertical draught of the ship is h feet. When the draught changes from 17·5 to 18·5 feet, what is the increased displacement of the vessel in cubic feet?

h	15	18	21
A	6020	6660	8250

(14)

11. The speed of a ship in knots (nautical miles per hour) has been noted at the following times :—

Speed	11·23	12·56	13·50	14·11	14·53
O'clock . . .	4	5	6	7	8

Plot on squared paper. What is the distance passed through during the hour after 6 o'clock? (14)

12. If $pu^{1\cdot0646} = 479$, find p when u is 3·25. (14)

13. The following corresponding values of x and y are given in a table :—

x	1·22	1·37	1·50
y	5·88	8·32	9·71

What is the probable value of x when y is 8? (10)

RESULTS

1. (*a*) 0·3015 ; (*b*) 4·5074, 1·5074, $\bar{1}$·5074, $\bar{3}$·5074 ; (*c*) 3·131 ; (*d*) £0·8875.

2. (*a*) 320 ; (*b*) 67 per cent., 26·3 per cent., 6·7 per cent. ; (*c*) 0·00029.

3. (*a*) $P\left(1 + \frac{r}{100}\right)^{12}$; (*b*) 8·24, 9 ; (*c*) $(x + \sqrt{10})(x - \sqrt{10})$; (*d*) 11,055 ft.

4. 2·1, 1·3.

5. Speed about 80 miles an hour between 0·2 hour and 0·3 hour ; speed about zero after 0·6 hour.

6. 4·47 miles ; $63\frac{1}{3}°$. **7.** 6750 horse-power. **8.** 0·09125 square inch.

9. $a = 1·1$, $b = 3·2$, $y = 13·275$. **10.** 6660 cubic feet.

11. 13·82 nautical miles. **12.** 136·6 **13.** 1·347.

THE END

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Not food for babes.
Considerable algebra given, before geom. is taken up.
Solid geometry very early